# LESSONS

ON

# PLANE TRIGONOMETRY,

AND ITS

APPLICATION TO THE MEASUREMENT

OF

## DISTANCES, HEIGHTS, AND AREAS,

AND TO

## NAVIGATION;

TOGETHER WITH

## THE ELEMENTARY FORMULÆ OF ANALYTICAL AND SOLID OR SPHERICAL TRIGONOMETRY.

BY EDWARD BUTLER, A.M., T.C.D.,

BARRISTER-AT-LAW,

ONE OF THE HEADS OF THE TRAINING ESTABLISHMENT, AND PROFESSOR OF MATHEMATICS AND NATURAL PHILOSOPHY, UNDER THE COMMISSIONERS OF NATIONAL EDUCATION IN IRELAND.

DUBLIN:

ALEXANDER THOM, PRINTER AND PUBLISHER,

87 & 88, ABBEY-STREET.

GROOMBRIDGE & SONS, 5, PATERNOSTER-ROW, LONDON.

1862.

# PREFACE.

The construction of a triangle or figure some of whose parts are given, is a problem for the solution of which directions can be deduced from the propositions of Geometry; and by the skilful use of instruments, the required figure, or one similar to it, can be represented on paper. Although these graphical constructions furnish a model or pattern of the figure, and show exactly the relative position of the parts given and the parts sought, they cannot be made available for ascertaining with any great accuracy the actual magnitude of the sides or angles; nor can the degree of approximation, or the amount of error, be in general estimated from these constructions.

It became, therefore, of importance to seek some method by which, from given parts of a triangle, the measures of the remaining parts might be obtained, either exactly or with a stated degree of approximation. Numerical computations, complying with these conditions, were therefore substituted for graphical constructions.

The branch of Mathematics having for its object the investigation of the principles and rules by which the several parts of a triangle are connected the one with the other, and by which their measures are accurately ascertained, is called *Trigonometry.*

The science of Trigonometry, however, now occupies a much wider field, and is employed in most departments of Mathematics pure and applied.

It is the object of the following lessons to place before the student in as simple a shape as possible, the fundamental principles and the more useful applications of Plane Trigonometry; the more important elementary formulæ of Analytical Trigonometry; and the leading properties on which depends the solution of Spherical Triangles.

The lessons on Plane Trigonometry require a knowledge of Arithmetic, the elements of Algebra, and the first two books only of Euclid's Elements; and those on Solid or Spherical Trigonometry suppose in addition an acquaintance with the first twenty-one propositions of the Eleventh Book. They have been thus framed to meet the case of many persons (particularly Teachers and Pupils of National Schools anxious to learn Navigation) to whom a well-grounded and practical knowledge of the subject would prove most serviceable in many ways, but who have had to forego this knowledge, because they had neither the leisure nor the opportunities required for learning the six books of Euclid, which in most treatises are assumed to be known by those who enter upon the study of Trigonometry.

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## THE GREEK ALPHABET.

| | | | | | |
|---|---|---|---|---|---|
| $\alpha$ | alpha. | $\iota$ | iōta. | $\rho$ | rhō. |
| $\beta$ | beeta. | $\kappa$ | kappa. | $\sigma$ | sigma. |
| $\gamma$ | gamma. | $\lambda$ | lambda. | $\tau$ | tau. |
| $\delta$ | delta. | $\mu$ | mu. | $\upsilon$ | yoopsee'lon. |
| $\epsilon$ | epsee'lon. | $\nu$ | nu. | $\phi$ | phee. |
| $\zeta$ | zeeta. | $\xi$ | ksee. | $\chi$ | chee (*ch*, hard). |
| $\eta$ | eeta. | $o$ | om'ecron. | $\psi$ | psee. |
| $\theta$ | theeta. | $\pi$ | pee. | $\omega$ | ō'mega. |

# LESSONS

ON

# PLANE TRIGONOMETRY.

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## INTRODUCTION.

**Measures of Magnitudes, Rational and Irrational Numbers—Preliminary Geometrical Propositions.**

### MEASURES OF MAGNITUDES, RATIONAL AND IRRATIONAL NUMBERS.

1. When a magnitude is compared with another of the same kind taken as standard, this standard or some ascertainable part of it may be contained an exact number of times in the given magnitude, or it may happen that no part however small of the standard can be contained exactly in it. In either case, the numerical expression by which the result of this comparison of a magnitude with the assumed standard, or unit of measure, is represented, is called the *measure* of the magnitude.

2. If the standard, or some submultiple of it, is contained an exact number of times in a given magnitude, this magnitude is said to be *rational* or *commensurable;* and its measure is expressed by a *rational*, or *commensurable*, *number*.

If, on the other hand, no submultiple whatever of the standard is contained an exact number of times in the given magnitude, the magnitude is said to be *incommensurable* or *irrational*, and its measure is an *incommensurable*, or *irrational*, *number*.

Thus, if the foot be taken as the unit of length, lines containing six feet, eight feet seven inches, fifteen feet and eighty-three-thousandths of a foot, will be commensurable, or rational, magnitudes; for they contain one foot, one-twelfth of a foot, one-thousandth of a foot, an exact number of times. The numbers 6, $8\frac{7}{12}$, $15{\cdot}083$ are the measures of the lines; they are commensurable, or rational, numbers.

But the diagonal of a square compared with its side is an irrational magnitude, and its measure, which is expressed

numerically by $\sqrt{2}$, is an irrational, or incommensurable, number. The same will be the case with the diagonal of every square whose side is a multiple or a submultiple of the unit of measure.

The circumference of a circle referred to its diameter as standard is also an incommensurable magnitude, and its measure, which is usually represented by the character $\pi$, and has for its value correct to the tenth decimal place, 3·1415926535, is an irrational number.

Again, if the foot square be taken for standard of superficial measurement, the area of the circle whose diameter is one foot in length, will be an incommensurable magnitude, and the numerical expression of its measure, ·78539 . . . . , will also be incommensurable.

3. The value of the numerical expression of an irrational magnitude is greater than the numbers which are measures of magnitudes less than the given one, but is less than the numbers which are measures of magnitudes greater than it.

Thus the value of the expression for the area of a circle whose diameter is one foot, is greater than ·7, ·78, ·785, ·7853, &c., &c., which are the measures of areas less than the circle. But the value is less than ·8, ·79, ·786, ·7854, &c., &c., which are the measures of areas greater than the circle.

Each of these numbers is, in square feet, an approximate value for the area of the circle.

In the same way, the value of the incommensurable number $\sqrt{2}$, which is the measure of the diagonal of a square when the side is taken as standard of length, is intermediate between the numbers

$$1,\ 1{\cdot}4,\ 1{\cdot}41,\ 1{\cdot}414,\ 1{\cdot}4142,$$

which are less than the diagonal, and the numbers

$$2,\ 1{\cdot}5,\ 1{\cdot}42,\ 1{\cdot}415,\ 1{\cdot}4143,$$

which are greater.

These numbers are all approximate values for the irrational number $\sqrt{2}$, those in the first set by defect, those in the second by excess; and as the differences between 1 and 2, 1·4 and 1·5, 1·41 and 1·42, 1·414 and 1·415, 1·4142 and 1·4143, (which are equal respectively to 1, ·1, ·01, ·001, ·0001), form a decreasing series of which ·1 is the ratio, it follows that either 1·4142 or 1·4143 will differ less from the correct value of $\sqrt{2}$, than any of the previous numbers; and the same will be true of 1·414 and 1·415, or of 1·41 and 1·42, and so on.

That is to say, the length of the diagonal can be expressed in commensurable numbers with a greater degree of approximation when the ten-thousandth part of the side is taken for

the term of comparison between it and the length of the side, than when the thousandth, or hundredth, or tenth, part is so used. And, again, if the hundred-thousandth or millionth part of the side be used as common measure between the diagonal and the side, the length of the diagonal may be expressed in commensurable numbers with a still greater degree of approximation.

4. In these and similar cases it is always possible to obtain for an incommensurable numerical expression a commensurable value that shall differ from the true value by a quantity less than any given quantity however small. For, let the value be calculated to $n$ decimal places, and let $a$ represent the number expressed by the digits obtained without taking account of the decimal point; then $\frac{a}{10^n}$ will represent the approximate value by defect, and $\frac{a+1}{10^n}$ the approximate value by excess. Either of these differs from the true value of the irrational number by a quantity less than the fraction $\frac{1}{10^n}$, and one of them by less than half this fraction.

Let the operation be carried out to $p$ more places, $a'$ representing the number expressed by the digits; then $\frac{a'}{10^{n+p}}$ and $\frac{a'+1}{10^{n+p}}$ will be approximate values by defect and by excess of the irrational number, from which each will differ by a quantity less than the fraction $\frac{1}{10^{n+p}}$, and one of them by less than half this fraction.

Now, the fraction $\frac{1}{10^{n+p}}$ is less than $\frac{1}{10^n}$; so that by increasing sufficiently the number of decimal places, a commensurable value may be found for the irrational number, such that it shall differ from the number by a quantity less than any assigned quantity however small.

5. The approximate value, whether by defect or by excess, that differs from the true value by a quantity less than half a unit of the $n$th order, is said to be the value of the irrational number to $n$ decimal places, and may be substituted for the magnitude itself without sensible error; or, at least, with one which can be appreciated.

6. When operations are to be performed with irrational

numbers, the result can be obtained within any requisite degree of approximation by using commensurable values sufficiently near to the given numbers.

Suppose that A is to be multiplied by the incommensurable number B, whose value is found to lie between $\frac{b}{10^n}$ and $\frac{b+1}{10^n}$. The product of A and B will fall between $\frac{Ab}{10^n}$ and $\frac{A(b+1)}{10^n}$, that is, between two numbers differing by the quantity $\frac{A}{10^n}$; and as $n$ may be taken sufficiently great to make this difference as small as may be required, it follows that the product of A and B can be obtained within any requisite degree of approximation.

If the quotient of A by B is to be found true to $n$ decimal places, it will be sufficient to calculate the quotient of $A \times 10^n$ by B to the unit's place, and mark off the required number of decimals.

7. If relations be proved concerning geometrical magnitudes, the same, when translated into algebraical, or arithmetical language, will hold concerning the measures of these magnitudes, whether they be expressed by rational or irrational numbers.

Thus, the XLVIIth Prop. of Book I. may be expressed arithmetically by stating, that in a right-angled triangle the sum of the second powers of the measures of the sides containing the right angle is equal to the second power of the measure of the hypotenuse.

In like manner the Vth Prop. of Book II. may be expressed arithmetically as follows:—"If a straight line be divided into two unequal, and also into two equal, parts, the product of the measures of the two unequal parts, together with the second power of the measure of the part between the points of section, shall be equal to the second power of the measure of half the line."

Hence, in geometrical relations concerning lines or surfaces, the measures of these lines or surfaces may be substituted without disturbing the relations.

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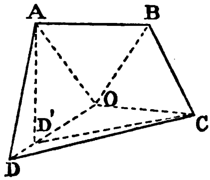
The term *ratio* in these lessons will signify the relation, not of the magnitudes themselves, but of their measures, and will be used in its arithmetical or algebraical sense to express that one quantity is a multiple, a part, or parts of another, as in Definitions 3, 4, and 5 of Book VII. of Euclid's Elements.

## PRELIMINARY GEOMETRICAL PROPOSITIONS.

8. *If four points be equally distant from a fifth, the opposite angles of the quadrilateral formed by joining these four points, shall be together equal to two right angles, and conversely.*

Let the four points, A, B, C, and D, be equally distant from a fifth point, O. Draw the straight lines, AB, BC, CD, DA, OA, OB, OC, OD.




Then BAO=ABO
DAO=ADO
DCO=CDO
BCO=CBO

Hence,

BAO+DAO+DCO+BCO=ABO+ADO+CDO+CBO

or, BAD+DCB=ADC+ABC

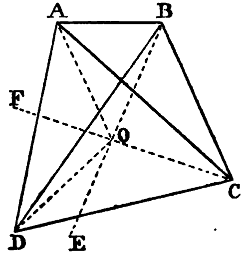
But the sum of these four angles is equal to four right angles; therefore, BAD+DCB, and ADC+ABC, are each equal to two right angles.

Conversely, *if in a quadrilateral the opposite angles be together equal to two right angles, a point can be found which shall be equally distant from the points of intersection of the sides.*

Let the angles B and D of the quadrilateral ABCD be together equal to two right angles, and O a point equally distant from A, B, and C. Draw the straight lines OA, OB, OC, and OD. These straight lines shall be equal.

For OA, OB, and OC, are equal by construction, and if OD be not equal to them, take a part OD′ equal to OA, or OB, or OC, and join A and D′, C and D′. Then because A, B, C, and D′, are equally distant from O, the angles ABC and AD′C are together equal to two right angles. And, by hypothesis, ABC and ADC are also equal to two right angles. Therefore, ADC and AD′C are equal, which is impossible. Therefore, OD is equal to OA, and OB, and OC. Therefore, the point O is equally distant from the four points A, B, C, and D.

9. *If four points be equally distant from a fifth, the angle contained by a diagonal and a side of the quadrilateral formed by joining the four points, shall be equal to the angle contained by the other diagonal and the side opposite to the former side, and conversely.*




Let A, B, C, and D, be four points equally distant from a fifth point, O. Draw the straight lines AB, BC, CD, DA, AC, BD, OA, OB, OC, and OD.

The angles ABD and ACD shall be equal.

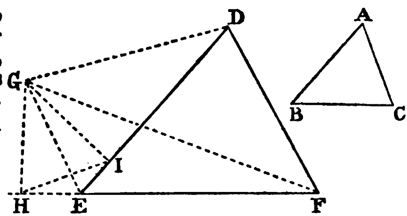
Produce BO to E, and CO to F:
then AOE=2 ABO,
DOE=2 DBO;
AOE − DOE=2 (ABO − DBO),
or, AOD=2 ABD.

Also, DOF=2 DCO, AOF=2 ACO;
DOF+AOF=2 (DCO+ACO), or, AOD=2ACD;
therefore, ABD=ACD.

It can be shown in a similar way that the angles, CBD and CAD, CAB and CDB, ACB and ADB, are equal.

The converse of this proposition is easily proved by an indirect demonstration.

10. *If two triangles be equiangular, and the sides containing a pair of equal angles be considered, the rectangle contained by two anhomologous sides, shall be equal to the rectangle contained by the two remaining anhomologous sides.**




Let ABC, DEF, be two equiangular triangles, having the angles A, B, and C, respectively equal to D, E, and F. Let the sides AB and BC, which contain the angle B of the one, and the sides DE and EF which contain the angle E of the other, be those under consideration: then AB.EF shall be equal to BC.DE.

Produce FE; through E draw the straight line EG, parallel to FD, and in it find the point G, so that GH be at right angles to FE, and equal to AB. From G draw GI at right angles to DE; join HI, GD, and GF.

Because the angles GHE and GIE are right angles, the four points G, H, E, and I, are equally distant from a fifth.

* In equiangular triangles, the sides opposite to equal angles are said to be homologous; and the sides not opposite to equal angles, to be anhomologous.

Therefore (9) the angles GIH and GEH are equal, as are also GHI and GEI. But GEH is equal to ACB, because both are equal to DFE; and GEI is equal to BAC, because both are equal to EDF. Therefore, the two triangles ABC and GHI are equal. Therefore, GI is equal to BC.

But the two triangles GEF and GED are equal; therefore the rectangle EF.GH, which is double of the triangle GEF, is equal to the rectangle DE.GI, which is double of the triangle GED. And because GH=AB, and GI=BC,

AB.EF=BC.DE.

It would be proved in the same way that

BC.FD=CA.EF, and CA.DE=AB.FD.

The proposition will be true if for the rectangles their measures be substituted. Therefore, if the measure of each side be represented by the small letter corresponding to the capital letter at the vertex of the angle opposite to it, the relations may be written

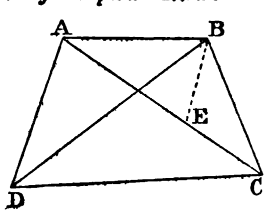
$$c\times d=a\times f,\ a\times e=b\times d,\ b\times f=c\times e.$$

From these, by division,

$$\frac{c}{a}=\frac{f}{d},\qquad \frac{a}{b}=\frac{d}{e},\qquad \frac{b}{c}=\frac{e}{f}.$$

That is to say that if two triangles be equiangular, the measures of their homologous sides shall be proportional.

11. *If four points be equally distant from a fifth, the rectangle contained by the diagonals of the quadrilateral formed by joining the four points shall be equal to the sum of the rectangles contained by the opposite sides of the quadrilateral.*




Let ABCD be the quadrilateral formed by joining the four points A, B, C, and D, which are equally distant from a fifth. Draw the diagonals AC and BD, and at the point B with the side BC make the angle CBE equal to ABD.

The two triangles ABD and EBC are equiangular, because they have (9) ADB=ECB, and ABD=EBC by construction.

Therefore (10) BD.CE=BC.AD.

The two triangles BDC and EAB are also equiangular, for they have (9) BDC=BAE, and DBC=DBE+EBC=DBE+DBA=EBA.

Therefore,

$$(10)\ BD.AE=CD.AB.$$

But

$$BD.CE+BD.AE=BD.AC.$$

Therefore,

$$BD.AC=BC.AD+CD.AB.$$

This proposition is commonly referred to as Ptolemy's Theorem or Lemma.

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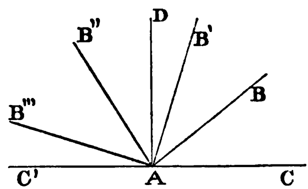
# LESSON I.

Angles, and their Measures—Angular Functions—Their reciprocal relations.

## ANGLES, AND THEIR MEASURES.

12. Let C′AC be a straight line, and AD a straight line drawn at right angles to it at the point A; and let another straight line, having one of its extremities at A, be supposed to turn on this point, as on a pivot, and to move in the same plane with AC and AD from the position of coincidence with AC to the position of coincidence with AD, and thence to the position of coincidence with AC′.




The different inclinations to the straight line AC which the revolving line has in its several positions form so many angles CAB, CAB′, CAD, CAB‴, increasing gradually in magnitude from acute to obtuse.

When the revolving line becomes coincident with AC′ it has strictly speaking no inclination to the straight line AC, and in this sense cannot be said to form any angle with it. As, however, a straight line that should move from the position of coincidence with AD to the position of coincidence with AC′ makes with the straight line AD a right angle, as well as a straight line moving from the position AC to the position AD makes a right angle with its initial position AC, it is customary in Mathematics to express the fact that a straight line is considered to have moved from its original position into one which is coincident with the continuation of the initial straight line, by stating that it then forms with the initial straight line an angle which is equal to two right angles.

It is in this sense that an angle is said to be equal to two right angles.

13. The right angle being a fixed magnitude, any angle will become quite determined when its relation to the right angle is known. The right angle is, therefore, taken as the standard, and the measure of any given angle is found by comparing it with the right angle.

For the purposes of measurement the right angle is divided into ninety equal parts called *degrees;* each degree into sixty equal parts called *minutes;* and each minute into sixty equal parts called *seconds*. So that the minute is the 5,400th part, and the second, the 324,000th part, of the right angle. If it be required to have a closer approximation to the ratio which an angle bears to the right angle than can be obtained by taking a second for common measure, decimal, not sexagesimal, subdivisions of the second are used.

The value of the ratio which an angle bears to the right angle—that is, the measure of the angle— is expressed in degrees, minutes, seconds, and decimal parts of a second.

Thus, if the angle CAB be the $\frac{3}{7}$ of a right angle, it will have for its measure, 38 degrees, 34 minutes, 17·13 seconds, which is written $38^\circ\ 34'\ 17''\cdot 13$.

When the measure of an angle is represented by a letter, it is not usual to annex any of the marks $^\circ$, $'$, $''$. The letters generally employed are the capital letters of the Roman alphabet and the small letters of the Greek alphabet.

The right angle is represented by $90^\circ$, and sometimes by the character $\frac{\pi}{2}$.

An angle equal to two right angles will be expressed by $180^\circ$, or by $\pi$.

14. When two angles taken together make up one right angle, they are said to be *complementary angles*, or one is the *complement* of the other.

Thus, if $A+B=90^\circ$, the angles A and B are complementary.

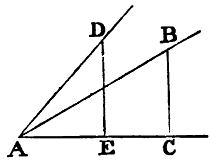
Again, an angle of $73^\circ\ 14'$ is the complement of, or has for its complement, an angle of $16^\circ\ 46'$.

15. When two angles taken together make up two right angles, they are said to be *supplementary angles*, or one is the *supplement* of the other.

Thus, if $\alpha+\beta=180^\circ$, the angles $\alpha$ and $\beta$ are supplementary.

Again, because $49^\circ\ 12'\ 37''$ and $130^\circ\ 47'\ 23''$ are equal to $180^\circ$, an angle of $49^\circ\ 12'\ 37''$ will be the supplement of, or will have for its supplement, an angle of $130^\circ\ 47'\ 23''$.

## ANGULAR FUNCTIONS.




16. If from any point B in one of the sides AB of the angle CAB, a straight line BC be drawn at right angles to the other side AC, the sides of the right angled triangle BAC thus formed, give rise, by the comparison of their measures two and two together, to the six following ratios:—

$$\frac{BC}{AB},\ \frac{BC}{AC},\ \frac{AB}{AC},$$
$$\frac{AB}{BC},\ \frac{AC}{BC},\ \frac{AC}{AB}.$$

As all the right-angled triangles that can be formed by the foregoing construction, wheresoever the point B is taken on either of the sides of the given angle, are equiangular, the ratios resulting from the comparison of the sides of each of these triangles will be equal to the ratios just set down.

These six ratios, whose values do not change when the angle CAB remains the same, are functions of the angle, and are called its *angular functions.*

They are also called *trigonometrical functions* and *circular functions.*

17. The ratio$\left(\frac{BC}{AB}\right)$ which the measure of the straight line (BC) drawn from a point (B) in one of the sides of an angle (CAB) at right angles to the other side, bears to the measure of the part (AB) of the first side intercepted between the perpendicular and the vertex (A) of the angle, is called the *sine* of the angle (CAB), and is represented by *sin.*

$$\text{Thus, } \frac{BC}{AB}=\sin CAB, \text{ and } \frac{DE}{AD}=\sin CAD.$$

18. The ratio $\left(\frac{BC}{AC}\right)$ which the measure of the straight line (BC) drawn from a point (B) in one of the sides of an angle (CAB) at right angles to the other side, bears to the measure of the part (AC) of this latter side intercepted between the perpendicular and the vertex (A) of the angle, is called the *tangent* of the angle (CAB), and is represented by *tan.*

$$\text{Thus, } \frac{BC}{AC}=\tan CAB, \text{ and } \frac{ED}{AE}=\tan CAD.$$

19. The ratio $\left(\frac{AB}{AC}\right)$ which the measure of the intercept (AB) between the vertex (A) of the angle (CAB) and a point (B) in one of the sides from which a straight line is drawn at right angles to the other side, has to the measure of the intercept (AC) on this side between the vertex (A) of the angle and the perpendicular, is called the *secant* of the angle CAB, and is represented by *sec*.

$$\text{Thus, } \frac{AB}{AC} = \sec CAB, \text{ and } \frac{AD}{AE} = \sec CAD.$$

20. The three remaining angular functions of the angle CAB,

$$\frac{AB}{BC}, \qquad \frac{AC}{BC}, \qquad \frac{AC}{AB},$$

are called respectively,

the *cosecant*, the *cotangent*, and the *cosine*

of the angle CAB, and are represented by

*cosec* *cot* *cos*

Thus,

$$\frac{AB}{BC} = \operatorname{cosec} CAB, \frac{AC}{BC} = \cot CAB, \text{ and } \frac{AC}{AB} = \cos CAB$$

$$\frac{AD}{DE} = \operatorname{cosec} CAD, \frac{AE}{DE} = \cot CAD, \text{ and } \frac{AE}{AD} = \cos CAD$$

The angular functions most used are the sine, the tangent, and the cosine.

### RECIPROCAL RELATIONS OF THE ANGULAR FUNCTIONS.

21. A comparison of the ratios which constitute the six angular functions shows that for any angle the sine and cosecant, the tangent and cotangent, the secant and cosine, are reciprocals.

Hence, A representing an angle,

$$\sin A = \frac{1}{\operatorname{cosec} A}, \tan A = \frac{1}{\cot A}, \sec A = \frac{1}{\cos A},$$

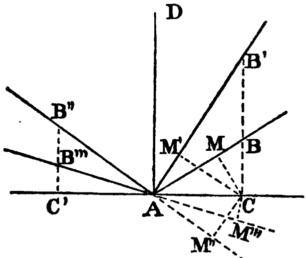
and

$$\cos A = \frac{1}{\sec A}, \cot A = \frac{1}{\tan A}, \operatorname{cosec} A = \frac{1}{\sin A}.$$

# LESSON II.

**Variation of the angular functions in Magnitude and in Sign—Complementary and Supplementary relations—Tabular Logarithms of angular functions.**

## VARIATION OF THE ANGULAR FUNCTIONS IN MAGNITUDE.




22. If a straight line be supposed to turn on the point A in the straight line C′C, from the position AC to the position AC′, the angle which it makes with the initial position AC will pass through all possible magnitudes from 0° to 180°. Let AB, AB′, AB″, AB‴ be some of its successive positions, and let AD be at right angles to AC.

From the points C and C′ draw straight lines at right angles to C′AC, meeting in B and B′ the straight lines AB and AB′, and in B″ and B‴ the straight lines AB″ and AB‴. From the point C draw the straight lines CM, CM′, CM″, CM‴ at right angles to the straight lines AB, AB′, AB″, AB‴.

Then,

$$(17)\ \sin CAB=\frac{CM}{AC},\ \sin CAB'=\frac{CM'}{AC},\ \sin CAB''=\frac{CM''}{AC},$$
$$\sin CAB'''=\frac{CM'''}{AC}.$$

$$(18)\ \tan CAB=\frac{BC}{AC},\ \tan CAB'=\frac{B'C}{AC},\ \tan CAB''=\frac{B''C'}{AC'},$$
$$\tan CAB'''=\frac{B'''C'}{AC'}.$$

$$(20)\ \cos CAB=\frac{AM}{AC},\ \cos CAB'=\frac{AM'}{AC},\ \cos CAB''=\frac{AM''}{AC},$$
$$\cos CAB'''=\frac{AM'''}{AC}.$$

23. While the straight line moves from AC towards AD, that is, when the angle increases from 0° towards 90°, the antecedents (CM, CM′) of the ratios for the sine increase, the consequents remaining the same: and therefore the numerical values of the ratios increase.

When the revolving line comes to coincide with AD, the antecedent and the consequent are equal, and the ratio has 1 for its value.

When the straight line moves from AD towards AC′, that is, when the angle continues to increase towards 180°, the antecedents decrease, the consequents remaining the same; the values of the ratios are, therefore, less and less.

When the moving line coincides with AC, or with AC′, that is, when the angles are respectively 0° or 180°, the ratios have no value, because their antecedents vanish. This is expressed by the following notation:—

$$\sin 0° = 0, \ \sin 180° = 0.$$

The symbol 0 is in this way used in mathematical formulæ to indicate that the quantities which are made equal to it can be decreased without limit; can be made smaller than any assignable quantity. In the case in point it signifies that the angle can be taken so small, or so near to 180°, that its sine shall be less than any given number, however small.

Hence it follows, that when the angle increases

$$\left.\begin{array}{l}\text{from } 0° \text{ to } 90° \\ \text{from } 90° \text{ to } 180°\end{array}\right\} \text{the sine} \left\{\begin{array}{l}\text{increases from 0 to 1} \\ \text{decreases from 1 to 0}\end{array}\right.$$

24. When the straight line moves from AC towards AD, the antecedents (BC, B′C) of the ratios for the tangent increase indefinitely in length, so that it will be always possible to find one that shall exceed any given length however great; therefore, the numerical values of the ratios increase without limit.

When the straight line coincides with AD, the value of the ratio, which has been increasing indefinitely while the revolving line was approaching indefinitely near to AD, becomes greater than any assignable number. It is under these circumstances said to become *infinite* at the limit, and this is expressed by writing $\tan 90° = \infty$.

This symbol $\infty$ is used in mathematical formulæ to indicate that the quantities which are said to be equal to it can be increased without limit; can be made to exceed any magnitude of the same kind however great.

If the straight line, after coinciding with AD, continues to move towards AC′, the antecedents will decrease, and consequently the numerical values of the ratios will also decrease.

When the straight line coincides with AC, or with AC′, the antecedents vanish, and there is no numerical value for the ratio. The notation by which this is represented is,

$$\tan 0° = 0, \ \tan 180° = 0.$$

It appears, then, from this discussion, that when an angle increases

from $0°$ to $90°$ } the tangent { increases from 0 to $\infty$
from $90°$ to $180°$ } the tangent { decreases from $\infty$ to 0

25. In the same way it will appear that when the angle increases

from $0°$ to $90°$ } the cosine { decreases from 1 to 0
from $90°$ to $180°$ } the cosine { increases from 0 to 1

The variations in magnitude of the other angular functions may be exhibited by a similar process.

### VARIATION OF THE ANGULAR FUNCTIONS IN SIGN.

26. In many mathematical investigations it is necessary to know not only the length of certain lines, but also their position with reference to one another, or with reference to certain lines or points which are regarded as fixed. For this purpose it is usual to prefix to the number expressing the measure of the line the sign +, or the sign —, according as the line is placed on one side or the other of certain determined parts of a diagram.

For instance, with reference to the two straight lines AC and AD, the lengths of straight lines (such as AC) measured in the direction AC to the right of AD have the sign + prefixed, and those measured on the opposite side of AD have the sign —.

The lengths of straight lines (such as CB, C′B″) measured in the direction AD, above the straight line AC, have the sign +, and those measured below that line would have the sign — prefixed.

A similar understanding holds with reference to the revolving line; and the straight lines CM, CM′, CM″, CM‴, will have the sign + prefixed to their measures, because they are all conceived to lie above the revolving line.

Again, the measures of AM and AM′ will have the sign +, because taken from the origin A on the revolving line itself, whereas the measures of AM″ and AM‴ will have the sign —, because taken, not on the revolving line, but on its prolongation.

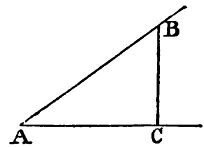
27. Hence, in order to give the angular functions their proper signs, it will be sufficient to examine the position, with reference to the origin and the initial line, of the straight lines which make up the several ratios.

For all angles not greater than one right angle, the angular functions will all have the sign +, because both the antecedents and the consequents are positive.

For all angles greater than a right angle, but not greater than two right angles, the sine will be positive, because both the antecedent and the consequent are positive; the tangent will be negative, because the antecedent is positive and the consequent negative; the cosine will be negative, because the antecedent is negative and the consequent positive. The secant and cotangent will also be negative, but the cosecant positive.

### COMPLEMENTARY AND SUPPLEMENTARY RELATIONS OF ANGULAR FUNCTIONS.

28. Let CAB be an angle, and CB a straight line at right angles to AC.




The two angles CAB, CBA are complementary, and if the former be represented by A, the latter will have for its expression 90° – A.

Now, with reference to the former of these angles, the ratios

$$\frac{BC}{AB}, \quad \frac{BC}{AC}, \quad \frac{AB}{AC}, \quad \frac{AC}{AB}, \quad \frac{AC}{BC}, \quad \frac{AB}{BC},$$

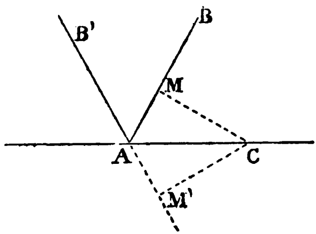
are the *sine*, *tangent*, *secant*, *cosine*, *cotangent*, *cosecant*, while with reference to the latter angle, they are

the *cosine*, *cotangent*, *cosecant*, *sine*, *tangent*, *secant*.

Hence the following complementary relations of the angular functions:

$$\sin A = \cos(90° - A), \ \tan A = \cot(90° - A),$$
$$\sec A = \operatorname{cosec}(90° - A),$$
$$\cos A = \sin(90° - A), \ \cot A = \tan(90° - A),$$
$$\operatorname{cosec} A = \sec(90° - A).$$

29. Let CAB be any angle A, and CAB′ its supplement 180° – A.




From the point C draw the straight lines CM, and CM′, at right angles to AB and to AB′ produced.

The two triangles CAM and CAM′ being equal, the angular functions of the two angles CAB and CAB′ are equal each to each in magnitude. But whereas the angular functions of the former are positive, the angle being less than a right angle, the sine and

cosecant alone of the latter are positive, while its other functions are negative.

If, on the other hand, the angle CAB′, greater than a right angle, be represented by A, and its supplement CAB by 180° − A, the sine and cosecant of A shall be positive, and its remaining angular functions negative, while all the angular functions of 180° − A shall be positive.

In either case, whether an angle A be acute or obtuse, the following are the supplementary relations of its angular functions:

$$\sin A = \sin(180° - A),\ \tan A = -\tan(180° - A),$$
$$\sec A = -\sec(180° - A),$$
$$\cos A = -\cos(180° - A),\ \cot A = -\cot(180° - A),$$
$$\text{cosec } A = \text{cosec}(180° - A).$$

## TABULAR LOGARITHMS OF ANGULAR FUNCTIONS.

30. From the preceding paragraphs it appears that if the values of the angular functions of all angles from 0° to 45° be known, those of the angular functions of all angles not exceeding 180° can be at once determined.

The values of the angular functions have been calculated with a sufficient degree of approximation for a series of angles increasing by 1′ from 0° to 45°, and have been registered in tables called "Tables of Natural Sines," &c., from which not only the value of an angular function of a given angle may be determined, but also the angle itself when one of its angular functions is a given number.

In most cases, however, computations are to be performed by means of logarithms. Tables available for this purpose have therefore been constructed; but only the mantissæ corresponding to the several angles are those of the logarithms to the base 10 of the numbers expressing the angular functions; whereas the registered characteristics are in all instances greater by 10 than those of the logarithms to the base 10.

Thus, sin 22° 30′ is ·3826834

logarithm to base 10 of sin 22° 30′ is $\bar{1}$·5828397

logarithm registered in tables for sin 22° 30′ is 9·5828397.

It will not, therefore, be sufficient to express a formula logarithmically in order to be able to make use of the tables; it will be necessary to take into account this difference between the logarithm of the value of an angular function and the logarithm entered in the tables as corresponding to it.

The registered logarithms may, for the sake of distinctness, be called the Tabular Logarithms of the angular functions, and will be indicated by the symbol L, while l will be used to indicate the logarithms taken to the base 10.

Then,

$$\text{l} \sin A = \text{L} \sin A - 10, \ \text{l} \tan A = \text{L} \tan A - 10,$$
$$\text{l} \sec A = \text{L} \sec A - 10,$$
$$\text{l} \cos A = \text{L} \cos A - 10, \ \text{l} \cot A = \text{L} \cot A - 10,$$
$$\text{l} \operatorname{cosec} A = \text{L} \operatorname{cosec} A - 10.$$

31. When a formula has been expressed logarithmically with reference to the base 10, the logarithms of the angular functions must be replaced by their values in terms of the corresponding Tabular Logarithms.

For instance, the formula $\sin A = \frac{b}{a \sin B}$ becomes when expressed logarithmically to the base 10

$$\text{l} \sin A = \text{l}b - \text{l}a - \text{l} \sin B$$

and when adapted for the tables

$$(\text{L} \sin A - 10) = \text{l}b - \text{l}a - (\text{L} \sin B - 10)$$

or

$$\text{L} \sin A = \text{l}b - \text{l}a - \text{L} \sin B + 20.$$

---

## LESSON III.

### Formulæ and Rules for the solution of Right-Angled Triangles.

32. In formulæ expressing relations connecting the several parts of a triangle, it is usual to represent the angles by the letters A, B, and C, and the sides opposite to these angles by the corresponding small letters $a$, $b$, and $c$.

In a right-angled triangle, the right angle will be the angle represented by C.

33. In the solution of right-angled triangles, four cases may be distinguished. There may be given

1. The two sides,
2. One side and an angle,
3. The hypotenuse and an angle,
4. The hypotenuse and a side;

and it may be required from these data to determine the remaining parts, or one or two of them.

**34. CASE I.—Given the two sides, to find the remaining parts.**

Here,
$$(18)\ \tan A=\frac{a}{b}$$

and
$$(17)\ \sin A=\frac{a}{c},\ \text{whence}\ c=\frac{a}{\sin A}.$$

Therefore,

$$\mathrm{l}\tan A=\mathrm{l}a-\mathrm{l}b,\qquad \text{and}\ \mathrm{l}c=\mathrm{l}a-\mathrm{l}\sin A;$$
$$(\mathrm{L}\tan A-10)=\mathrm{l}a-\mathrm{l}b,\qquad \text{and}\ \mathrm{l}c=\mathrm{l}a-(\mathrm{L}\sin A-10),$$
or
$$\mathrm{L}\tan A=10+\mathrm{l}a-\mathrm{l}b,\ \text{and}\ \mathrm{l}c=10+\mathrm{l}a-\mathrm{L}\sin A.$$
$$B=90^\circ-A.$$

RULE.—To find one of the angles, add 10 to the logarithm of the side opposite to the required angle, and from the sum take the logarithm of the other side; the result will be the tabular logarithm of the tangent of the angle.

When one of the angles is found, take its measure from 90°, and the difference will be the second angle.

To find the hypotenuse, add 10 to the logarithm of one of the given sides, and from the sum subtract the tabular logarithm of the sine of the angle opposite to it; the result will be the logarithm of the hypotenuse.

EXAMPLE I.—Given $a=111{\cdot}45$, and $b=247{\cdot}58$ chains.

*Computation for A.*

| | |
|---|---|
| 10+l. 111·45, . . . | 12·0460801 |
| —l. 247·58, . . . | 2·3937156 |
| L tan A, . . | 9·6523645 |
| L tan 24° 11′, . | 9·6523123 |
| 9″, | 522 |

A=24° 11′ 9″.

*Computation for c.*

| | |
|---|---|
| 10+l. 111·45, . . . | 12·0460801 |
| —L sin 24° 11′, . . | 9·6124211 |
| 9″, . | 422 |
| lc, . . . . | 2·4336168 |

c=271·4, chains.

B=65° 48′ 51″.

---

EXERCISES.

1. Given $a=543{\cdot}92$ feet, and $b=226{\cdot}63$ feet, to calculate the hypotenuse and the angles.

2. Given $a=3$ and $b=6$ in a right-angled triangle, to find the angles.

3. To compute the angles and hypotenuse in a right-angled triangle whose sides are 47·19 and 12·177 chains.

4. Given in a right-angled triangle $a=649{\cdot}844$ and $b=45{\cdot}432$; to find the hypotenuse and angles.

5. If the sides of a right-angled triangle are equal respectively to 162·63 and 360·651 chains, what will be the angles and the hypotenuse?

35. CASE II.—Given a side and an angle, to find the remaining parts.

1. Let A be the given angle, and $a$ the given side.

Then, $$B=90^\circ-A$$

$$(17)\ \frac{a}{c}=\sin A;\ \text{whence } c=\frac{a}{\sin A}$$

$$(18)\ \frac{a}{b}=\tan A;\ \text{whence } b=\frac{a}{\tan A.}$$

Therefore,

$$lc=la-l\sin A,\qquad \text{and } lb=la-l\tan A;$$

whence $lc=la-(L\sin A-10)$, and $lb=la-(L\tan A-10)$,

or $lc=10+la-L\sin A$, and $lb=10+la-L\tan A$.

RULE.—When an angle and the opposite side are given, to find the hypotenuse, add 10 to the logarithm of the side, and from the result subtract the tabular logarithm of the sine of the angle; the difference will be the logarithm of the hypotenuse.

To find the other side, add 10 to the logarithm of the given side, and from the result subtract the tabular logarithm of the tangent of the angle; the difference will be the logarithm of the required side.

2. Let A be the given angle, and $b$ the given side.

Then, $$B=90^\circ-A$$

$$(20)\ \frac{b}{c}=\cos A;\ \text{whence } c=\frac{b}{\cos A}$$

$$(18)\ \frac{a}{b}=\tan A;\ \text{whence } a=b\tan A.$$

When expressed logarithmically, these formulæ become

$$lc=lb-l\cos A,\qquad \text{and } la=lb+l\tan A,$$

or $lc=lb-(L\cos A-10)$, and $la=lb+(L\tan A-10)$,

that is,

$$lc=10+lb-L\cos A,\qquad \text{and } la=lb+L\tan A-10.$$

RULE.—When an angle and the adjacent side are given, to find the hypotenuse, add 10 to the logarithm of the side, and from the result subtract the tabular logarithm of the cosine of the angle; the difference will be the logarithm of the hypotenuse.

To find the side opposite to the given angle, add to the logarithm of the given side the tabular logarithm of the tangent of the angle, and from the result subtract 10; the remainder will be the logarithm of the required side.

EXAMPLE II.—Given in a right-angled triangle A=68° 42′, and $a$=4837 feet, to find the hypotenuse and the other side.

| *Computation for c.* | |
|---|---|
| 10+l. 4837, . . . | 13·6845761 |
| —L sin 68° 42′, . . | 9·9692720 |
| l$c$, . . . . | 3·7153041 |
| $c$=5191·65. | |

| *Computation for b.* | |
|---|---|
| 10+l. 4837, . . . | 13·6845761 |
| —L tan 68° 42′, . . | 10·4090649 |
| l$b$, . . . . | 3·2755112 |
| $b$=1885·87. | |

EXAMPLE III.—Given A=48° 13′ 28″, and $b$=384·9 feet, to calculate the hypotenuse and the side $a$.

| *Computation for c.* | |
|---|---|
| 10+l. 384·9, . . . | 12·5853479 |
| —L cos 48° 14′, . . | 9·8235386 |
| 32″, . . | 754 |
| l$c$, . . . . | 2·7617339 |
| $c$=577·74. | |

| *Computation for a.* | |
|---|---|
| l. 384·9, . . . . | 2·5853479 |
| +L tan 48° 13′, . . | 10·0488666 |
| 28″, . | 1187 |
| —10 | |
| l$b$, . . . . | 2·6343332 |
| $b$=430·85. | |

## EXERCISES.

6. In a right-angled triangle the angle A is 39° 27′ 43″, and the opposite side is 13·8 chains; compute the remaining parts.

7. If A=29° 43′, and $a$=486·82 feet, find the remaining parts of the right-angled triangle.

8. Given B=19° 12′ 15″, and $a$=15·37 feet, to find the hypotenuse and the remaining side.

9. Given A=49° 47′ 5″, and $b$=8374·5 feet, to compute the remaining parts of the right-angled triangle.

10. In a right-angled triangle, given B=57° 58′ 36″, and $b$=27 feet, to find the remaining parts.

11. Given an angle of a right-angled triangle, 73° 55′, and the side adjacent to it, 1537 inches, to compute the hypotenuse and the other side.

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36. CASE III.—Given the hypotenuse and an angle, to find the sides.

Let A be the given angle, then B=90°—A.

$$(17)\quad \frac{a}{c}=\sin A, \text{ whence } a=c\sin A.$$

$$(20)\quad \frac{b}{c}=\cos A, \text{ whence } b=c\cos A.$$

Hence the logarithmic formulæ are,

$$la=lc+l\sin A, \qquad \text{and } lb=lc+l\cos A,$$

or, $$la=lc+(L\sin A-10), \text{ and } lb=lc+(L\cos A-10),$$

that is,

$$la=lc+L\sin A-10, \qquad \text{and } lb=lc+L\cos A-10.$$

RULE.—To find the side opposite to the given angle, add to the logarithm of the hypotenuse the tabular logarithm of the sine of the given angle, and take 10 from the result; the remainder is the logarithm of the side.

To find the side adjacent to the given angle, add to the logarithm of the hypotenuse the tabular logarithm of the cosine of the angle, and take 10 from the result; the remainder will be the logarithm of the required side.

EXAMPLE IV.—Given $c=678{\cdot}54$ feet, and $B=35°\ 12'\ 34''$, to find the remaining parts of the right-angled triangle.

| *Computation for a.* | | *Computation for b.* | |
|---|---|---|---|
| l. 678·54, | 2·8315755 | l. 678·54, | 2·8315755 |
| + L cos 35° 13′, | 9·9122099 | + L sin 35° 12′, | 9·7607483 |
| 26″, | 386 | 34″, | 1015 |
| —10 | | —10 | |
| l$a$, | 2·7438240 | l$b$, | 2·5924253 |
| $a=554{\cdot}4$ feet. | | $b=391{\cdot}22$ feet. | |

$A=54°\ 47'\ 26''$.

---

EXERCISES.

12. Given in a right-angled triangle the hypotenuse, 291 feet, and one of the angles, 47° 55, to find the remaining parts.

13. Given $c=3245$ chains, and $B=57°\ 13'$, find the sides of the right-angled triangle.

14. Given $c=21{\cdot}543$ chains, and $A=61°\ 40'\ 15''$, find the sides of the right-angled triangle.

15. When the hypotenuse is 15·23 chains, and one of the angles 8° 37′, what is the length of each of the two sides?

16. In a right-angled triangle one of the angles is 75° 14′ 48″, and the hypotenuse is $\sqrt{49{\cdot}231}$; compute the two sides.

---

37. Case IV.—Given the hypotenuse and one side, to find the angles and the remaining side.

Let $a$ be the given side.

Then, (17) $\sin A=\dfrac{a}{c}$,

(20) $\cos A=\dfrac{b}{c}$, whence $b=c\cos A$.

Then, $\text{l}\sin A=\text{l}a-\text{l}c$, and $\text{l}b=\text{l}c+\text{l}\cos A$,

$(\text{L}\sin A-10)=\text{l}a-\text{l}c$, and $\text{l}b=\text{l}c+(\text{L}\cos A-10)$;

or, $\text{L}\sin A=10+\text{l}a-\text{l}c$, and $\text{l}b=\text{l}c+\text{L}\cos A-10$,

$B=90°-A$.

RULE.—To find the angle opposite to the given side, add 10 to the logarithm of this side, and from the sum subtract the logarithm of the hypotenuse; the result will be the tabular logarithm of the sine of the required angle.

To find the second side, to the logarithm of the hypotenuse add the tabular logarithm of the cosine of the given angle, and diminish the sum by 10; the result will be the logarithm of the required side.

When the angle opposite to the given side is found, the other angle will be the difference between it and 90°.

EXAMPLE V.—In a right-angled triangle the hypotenuse is 371·43 feet, and one of the sides is 183·5 feet. Required the remaining parts.

| *Computation for A.* | | *Computation for b.* | |
|---|---|---|---|
| 10+l. 183·5, | 12·2636361 | l. 371·43, | 2·5698770 |
| —l. 371·43, | 2·5698770 | +L cos 29° 37′ | 9·9391953 |
| | | 38″ | 455 |
| L sin A, | 9·6937591 | —10 | |
| L sin 29° 36′, | 9·6936758 | l*b*, | 2·5091178 |
| 22″, | 833 | *b*=322·93 feet. | |
| A=29° 36′ 22″. | | | |

B=60° 23′ 38″.

## EXERCISES.

17. Given the hypotenuse 15·6, and one side 4·3 furlongs; find the angles and the second side.

18. Given $c=4978{\cdot}79$ feet, and $a=785{\cdot}39$ feet, to compute the side $b$ and the angles.

19. In a right-angled triangle, given $b=4{\cdot}875$ and $c=10{\cdot}45$, to find the other side and the angles.

20. Find the angles of a right-angled triangle in which the hypotenuse measures 32·45 chains, and one of the sides 17·57 chains.

The student may now pass on to the Lesson on Navigation; for no more of Trigonometry than is to be learned in the three foregoing lessons is needed for the solution of problems belonging to *Plane*, *Parallel*, *Middle Latitude*, or *Mercator's Sailing*.

# LESSON IV.

Relations connecting the angular functions of an angle—Relations connecting the angular functions of an angle with those of its half—Examples and Exercises.

## RELATIONS CONNECTING THE ANGULAR FUNCTIONS OF AN ANGLE.

38. *The tangent of an angle is equal to the sine divided by the cosine of the angle.*

Let CAB (28) be an angle less than a right angle, and let it be represented by A. Draw the straight line CB at right angles to AC.

Then,

$$(18)\ \tan A=\frac{BC}{AC},\ (17)\ \sin A=\frac{BC}{AB},\ (20)\ \cos A=\frac{AC}{AB}.$$

But

$$\frac{BC}{AC}=\frac{BC}{AB}\div\frac{AC}{AB}.$$

Therefore,

$$\tan A=\frac{\sin A}{\cos A}.$$

If the given angle be greater than a right angle, and represented by A′, then its supplement, 180°—A′, will be less than a right angle, and from what has just been proved,

$$\tan(180°-A')=\frac{\sin(180°-A')}{\cos(180°-A')}.$$

But

$$(29)\ \tan(180°-A')=-\tan(A'),\ \sin(180°-A')=\sin A',$$
$$\text{and } \cos(180°-A')=-\cos A'.$$

Therefore,

$$-\tan A'=\frac{\sin A'}{-\cos A'},\ \text{or}\ \tan A'=\frac{\sin A'}{\cos A'}.$$

The theorem is, therefore, true for all angles that do not exceed two right angles.

39. *The sum of the squares of the sine and cosine of an angle is equal to* 1.

The right-angled triangle ACB (28) gives $BC^2+AC^2=AB^2$: substituting for the geometrical magnitudes their measures, and dividing both sides of the equation by $AB^2$, the following relation is obtained:

$$\frac{BC^2}{AB^2}+\frac{AC^2}{AB^2}=1,\text{or}\ \left(\frac{BC}{AB}\right)^2+\left(\frac{AC}{AB}\right)^2=1.$$

Therefore, when A is less than a right angle,

$$\sin^2 A + \cos^2 A = 1.*$$

If the given angle A′ be greater than a right angle, $180° - A'$ will be less than a right angle, and therefore,

$$\sin^2(180° - A') + \cos^2(180° - A') = 1.$$

But $\sin(180° - A') = \sin A'$, $\cos(180° - A') = -\cos A'$. Therefore,

$$\sin^2 A' + \cos^2 A' = 1.$$

The theorem is true for all angles that do not exceed two right angles.

40. *The square of the secant of an angle exceeds by* 1 *the square of the tangent of the angle.*

If in the relation $BC^2 + AC^2 = AB^2$, derived from the right-angled triangle ACB (28), the geometrical magnitudes be replaced by their measures, and both sides be divided by $AC^2$, it becomes

$$\frac{AB^2}{AC^2} = 1 + \frac{BC^2}{AC^2} \text{ or } \left(\frac{AB}{AC}\right)^2 = 1 + \left(\frac{BC}{AC}\right)^2.$$

Hence, when the angle CAB or A is less than a right angle,

$$\sec^2 A = 1 + \tan^2 A.$$

If the given angle A′ be greater than a right angle, but less than two right angles, then $180° - A'$ will be less than a right angle, and the preceding formula holds for this angle, that is to say,

$$\sec^2(180° - A') = 1 + \tan^2(180° - A').$$

But $\sec(180° - A') = -\sec A'$, $\tan(180° - A') = -\tan A'$. Therefore,

$$\sec^2 A' = 1 + \tan^2 A'.$$

The theorem holds for angles not greater than two right angles.

The relation $\sec^2 A = 1 + \tan^2 A$ can be derived from the relation (39). Dividing by $\cos^2 A$ both sides of the equation

$$\sin^2 A + \cos^2 A = 1$$

the result will be $\dfrac{\sin^2 A}{\cos^2 A} + 1 = \dfrac{1}{\cos^2 A}.$

But (38) $\dfrac{\sin A}{\cos A} = \tan A$, and (21) $\dfrac{1}{\cos A} = \sec A$.

Therefore, $\tan^2 A + 1 = \sec^2 A.$

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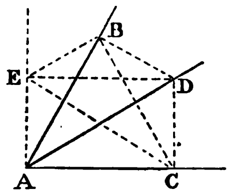
* The power of an angular function is expressed by placing the index between the name of the function and the measure of the angle.

## RELATIONS CONNECTING THE ANGULAR FUNCTIONS OF AN ANGLE AND THOSE OF ITS HALF.

41. *The sine of an angle is equal to twice the product of the sine of half the angle by the cosine of half the angle.*




Let BAC be an angle less than a right angle, and represented by A. And let this angle be bisected by the straight line AD.

Through A draw the straight line AE at right angles to AC; through any point D in the straight line AD, draw DC, DB, DE, at right angles to AC, AB, and AE. Draw the straight lines BC, BE, and EC.

Because the angles ABD and ACD are right angles, the four points A, C, D, B, are equally distant from a fifth (8); and so are the four points A, D, B, E (9), because the angles AED and ABD are equal. Hence the five points A, C, D, B, E, are equally distant from the same point, and consequently the angle BEC is equal to the given angle BAC (9).

The quadrilateral ACDB gives (11)

$$BC.AD = DC.AB + DB.AC.$$

Or, because by construction DB=DC, and AC=AB,

$$BC.AD = 2\, DC.AB.$$

By substituting for the surfaces their measures, and then dividing by $AD^2$, this equation becomes,

$$\frac{BC}{AD} = 2\frac{DC}{AD}\cdot\frac{AB}{AD}.$$

But, $$\frac{BC}{AD} = \frac{BC}{EC} = \sin BEC = \sin A$$

$$\frac{DC}{AD} = \sin DAC = \sin \tfrac{1}{2}A, \text{ and } \frac{AB}{AD} = \cos BAD = \cos \tfrac{1}{2}A.$$

Hence, $$\sin A = 2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A.$$

When the given angle A′ exceeds a right angle, then $180^\circ - A'$ is less than a right angle, and, therefore,

$$\sin(180^\circ - A') = 2 \sin \tfrac{1}{2}(180^\circ - A') \cos \tfrac{1}{2}(180^\circ - A'),$$

that is,

$$\sin(180^\circ - A') = 2 \sin(90^\circ - \tfrac{1}{2}A') \cos(90^\circ - \tfrac{1}{2}A'),$$

and as (29) $\sin(180^\circ - A') = \sin A'$, the foregoing expression becomes,

$$\sin A' = 2 \cos \tfrac{1}{2}A' \sin \tfrac{1}{2}A'.$$

42. *The cosine of an angle is equal to the difference between the squares of the cosine and sine of half the angle.*

The same construction being made as in (41), the quadrilateral ADBE gives,

$$AB.ED=BD.AE+AD.BE, \text{ or, } AB^2=BD^2+AD.BE,$$

because, by construction, AB=ED, and BD=AE.

After replacing the geometrical magnitudes by their measures, and dividing both sides by $AD^2$, the last relation becomes,

$$\frac{AB^2}{AD^2}=\frac{BD^2}{AD^2}+\frac{BE}{AD}, \text{ or } \left(\frac{AB}{AD}\right)^2=\left(\frac{BD}{AD}\right)^2+\left(\frac{BE}{AD}\right)^2.$$

But, $\frac{AB}{AD}=\cos BAD=\cos\frac{1}{2}A$, $\frac{BD}{AD}=\sin BAD=\sin\frac{1}{2}A$, and

$$\frac{BE}{AD}=\frac{BE}{EC}=\cos BEC=\cos A.$$

Hence, $\cos^2\frac{1}{2}A=\sin^2\frac{1}{2}A+\cos A$,

and, therefore, $\cos A=\cos^2\frac{1}{2}A-\sin^2\frac{1}{2}A$.

If the angle A′ lies between 90° and 180°, the formula will hold for its supplement 180°—A′; therefore,

$$\cos(180^\circ-A')=\cos^2\tfrac{1}{2}(180^\circ-A')-\sin^2\tfrac{1}{2}(180^\circ-A')$$
$$=\cos^2(90^\circ-\tfrac{1}{2}A')-\sin^2(90^\circ-\tfrac{1}{2}A').$$

Now, (29) $\cos(180^\circ-A')=-\cos A'$;

and (28) $\cos(90^\circ-\frac{1}{2}A')=\sin\frac{1}{2}A'$, $\sin(90^\circ-\frac{1}{2}A')=\cos\frac{1}{2}A'$;

therefore, $-\cos A'=\sin^2\frac{1}{2}A'-\cos^2\frac{1}{2}A'$,

or, changing the signs on both sides of the equation,

$$\cos A'=\cos^2\tfrac{1}{2}A'-\sin^2\tfrac{1}{2}A'.$$

43. *The tangent of an angle is equal to the quotient of twice the tangent of half the angle, by the difference between* 1 *and the square of the tangent of half the angle.*

The same construction being made as in (41), the two quadrilaterals ACDB and ADBE give

$$BC.AD=2\,DB.AB$$
$$BE.AD=AB^2-BD^2.$$

Replacing the surfaces by their measures, and dividing the corresponding sides of these equations the one by the other, the result will be

$$\frac{BC}{BE}=\frac{2\,DB.AB}{AB^2-BD^2}, \text{ whence } \frac{BC}{BE}=\frac{2\frac{DB}{AB}}{1-\left(\frac{BD}{AB}\right)^2},$$

after dividing by $AB^2$ both terms of the second fraction.

Now, $\frac{BC}{BE}$=tan BEC=tan A, $\frac{DB}{AB}$=tan DAB=tan $\frac{1}{2}$A.

Therefore, $$\tan A=\frac{2\tan\frac{1}{2}A}{1-\tan^2\frac{1}{2}A}.$$

That the theorem holds when the given angle exceeds a right angle may be shown as follows:

Let A′ be an angle greater than a right angle, then the angle 180°—A′ will be less than a right angle, and, therefore,

$$\tan(180°-A')=\frac{2\tan\frac{1}{2}(180°-A')}{1-\tan^2\frac{1}{2}(180°-A')}$$
$$=\frac{2\tan(90°-\frac{1}{2}A')}{1-\tan^2(90°-\frac{1}{2}A')}.$$

But (29) $\tan(180°-A')=-\tan A'$,

and (28) $\tan(90°-\frac{1}{2}A')=\cot\frac{1}{2}A'$.

Therefore, $$-\tan A'=\frac{2\cot\frac{1}{2}A'}{1-\cot^2\frac{1}{2}A'},$$

or, multiplying both terms of the fraction by $\tan^2\frac{1}{2}A'$, and bearing in mind that

(21) $\tan\frac{1}{2}A'\cot\frac{1}{2}A'=1$, $\tan^2\frac{1}{2}A'\cot^2\frac{1}{2}A'=1$,

$$-\tan A'=\frac{2\tan\frac{1}{2}A'}{\tan^2\frac{1}{2}A'-1}, \text{ whence } \tan A'=\frac{2\tan\frac{1}{2}A'}{1-\tan^2\frac{1}{2}A'}.$$

This theorem may be derived from the two preceding theorems.

For (41) $\sin A=2\sin\frac{1}{2}A\cos\frac{1}{2}A$

(42) $\cos A=\cos^2\frac{1}{2}A-\sin^2\frac{1}{2}A$.

Dividing the one by the other the corresponding sides of these equations, the result will be

$$\frac{\sin A}{\cos A}=\frac{2\sin\frac{1}{2}A\cos\frac{1}{2}A}{\cos^2\frac{1}{2}A-\sin^2\frac{1}{2}A}=\frac{2\frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A}}{1-\frac{\sin^2\frac{1}{2}A}{\cos^2\frac{1}{2}A}}.$$

Hence, $$\tan A=\frac{2\tan\frac{1}{2}A}{1-\tan^2\frac{1}{2}A}.$$

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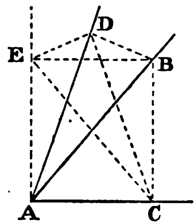
## EXAMPLES AND EXERCISES.

Some of these examples and exercises require an acquaintance with quadratic equations, and many of them depend for their solution upon one or more of the others. If the student, in working them out for the first time, should not see his way to the solution of an exercise after a few attempts, let him pass on to others, and afterwards retrace his steps, and try again the exercises he has failed to solve.

## EXAMPLES.




EXAMPLE I.—Prove that if the angles A, B, and A+B, be respectively less than one right angle,

$\sin(A+B)=\sin A \cos B+\sin B \cos A.$

Let CAB be the angle A, and BAD the angle B. From the point A in the straight line AC draw AE at right angles to it, and from any point B, in the straight line AB, draw the straight lines BC, BD, BE, at right angles to AC, AD, and AE. Join the points C and D, C and E, D and E.

Then, BC=AE, BE=AC, BA=EC; and because the points A, C, B, D, and E are equally distant from the same point, the angle CDE is a right angle, and the angle DEC is equal to the angle DAC or (A+B).

The quadrilateral ACBD gives

$$AB.DC=BC.AD+BD.AC;$$

whence,

$$\frac{DC}{EC}=\frac{BC}{AB}\cdot\frac{AD}{AB}+\frac{BD}{AB}\cdot\frac{AC}{AB};$$

that is, $\sin(A+B)=\sin A \cos B+\sin B \cos A.$

In the foregoing construction, each of the angles A, B, and A+B, has been supposed less than a right angle; but the theorem will be proved by a similar process if one or two of them be greater than one right angle, but less than two.

EXAMPLE II.—Find expressions for tan A$\pm$ cot A in function of 2A.

I. $$\tan A+\cot A=\tan A+\frac{1}{\tan A}$$
$$=\frac{1+\tan^2 A}{\tan A}$$
$$=\frac{\sec^2 A}{\tan A}$$
$$=\frac{1}{\cos^2 A \tan A}$$
$$=\frac{2}{2\sin A\cos A}$$
$$=\frac{2}{\sin 2A}, \text{ the required expression.}$$

II. $$\tan A-\cot A=\frac{1}{\cot A}-\cot A.$$
$$=\frac{1-\cot^2 A}{\cot A}.$$
$$=2\,\frac{1-\cot^2 A}{2\cot A}$$
$$=-2\cot 2A, \text{ the required expression.}$$

EXAMPLE III.—Find the numerical value for the sine of 150°.

$$\sin 150° = \sin(180° - 150°) = \sin 30°.$$

Let $a$ be the side of an equilateral triangle; let one of the angles A be bisected, and the straight line be produced to meet the opposite side. Then, $\sin \frac{1}{2}A = \frac{\frac{1}{2}a}{a} = \frac{1}{2}$. But the angle of an equilateral triangle has 60° for its measure; therefore,

$$\sin 30° = \sin 150° = \tfrac{1}{2}.$$

EXAMPLE IV.—If $1 + \sin A = 2\cos^2(A - \frac{1}{2}a)$, what is the value of the angle A, $a$ being a given angle?

Because, $\quad 2\cos^2(A - \frac{1}{2}a) = 1 + \cos(2A - a)$;

then, $\quad 1 + \sin A = 1 + \cos(2A - a)$, or, $\sin A = \cos(2A - a)$.

Therefore, the angles A and $(2A - a)$ are complements, and

$$A = 90° - (2A - a), \text{ or, } A = 30° + \tfrac{1}{3}a.$$

EXAMPLE V.—What value of $\phi$ will make the expression $\sin\phi\cos\phi$ a maximum?

The expression $\sin\phi\cos\phi$, it is clear, will be a maximum if $2\sin\phi\cos\phi$ be a maximum; that is, if $\sin 2\phi$ be a maximum. But the greatest value for a sine is 1, and this is the sine of a right angle. Hence, $\sin 2\phi$ will be a maximum, if $2\phi$ be a right angle, that is, if $\phi$ be half a right angle.

Therefore, the expression $\sin\phi\cos\phi$ is a maximum when the angle $\phi$ is half a right angle.

EXAMPLE VI.—If $\tan^3\theta = \frac{b}{a}$, then $\frac{a}{\cos\theta} + \frac{b}{\sin\theta} = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

For, $$\frac{a}{\cos\theta} + \frac{b}{\sin\theta} = a\left(\frac{1}{\cos\theta} + \frac{\frac{b}{a}}{\sin\theta}\right) = a\left(\frac{1}{\cos\theta} + \frac{\tan^3\theta}{\sin\theta}\right)$$
$$= a\left(\frac{1}{\cos\theta} + \frac{\sin^2\theta}{\cos^3\theta}\right) = a\left(\frac{\cos^2\theta + \sin^2\theta}{\cos^3\theta}\right)$$
$$= a\sec^3\theta.$$

Because $\tan^3\theta = \frac{b}{a}$, or $\tan\theta = \frac{b^{\frac{1}{3}}}{a^{\frac{1}{3}}}$, and $\sec^2\theta = 1 + \tan^2\theta$,

$$\sec\theta = (1 + \tan^2\theta)^{\frac{1}{2}} = \left(1 + \frac{b^{\frac{2}{3}}}{a^{\frac{2}{3}}}\right)^{\frac{1}{2}},$$

and $$\sec^3\theta = \left(1 + \frac{b^{\frac{2}{3}}}{a^{\frac{2}{3}}}\right)^{\frac{3}{2}}.$$

Therefore, $$\frac{a}{\cos\theta} + \frac{b}{\sin\theta} = a\left(1 + \frac{b^{\frac{2}{3}}}{a^{\frac{2}{3}}}\right)^{\frac{3}{2}} = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}.$$

EXAMPLE VII.—What are the values of $\sin A$ and $\cos A$ that satisfy the equation $m\sin A + n\cos A = a$.

From the given equation,

$$\cos A=\frac{a-m\sin A}{n}, \text{ and } \cos^2 A=\frac{(a-m\sin A)^2}{n^2};$$

hence, $$\sin^2 A+\frac{a^2-2am\sin A+m^2\sin^2 A}{n^2}=1;$$

or, $$(n^2+m^2)\sin^2 A-2am\sin A+a^2-n^2=0;$$

whence, $$\sin A=\frac{am\pm n\sqrt{(m^2+n^2-a^2)}}{m^2+n^2}.$$

By a similar process,

$$\cos A=\frac{an\pm m\sqrt{(m^2+n^2-a^2)}}{m^2+n^2}.$$

EXAMPLE VIII.—Adapt to logarithmic computation the expression $a+b$, or $a-b$.*

Let $x=a+b$, and $y=a-b$.

Then, $x=a\left(1+\frac{b}{a}\right)$ and $y=a\left(1-\frac{b}{a}\right)$,

or, $x=a(1+\tan\phi)$ $y=a(1-\tan\phi)$,

by taking an angle $\phi$ whose tangent is equal to $\frac{b}{a}$; which will always be possible, as the tangents vary in magnitude from 0 to $\infty$.

But,

$$1+\tan\phi=\sqrt{2}\frac{\sin(45^\circ+\phi)}{\cos\phi}, \text{ and } 1-\tan\phi=\sqrt{2}\frac{\sin(45^\circ-\phi)}{\cos\phi}.$$

Therefore,

$$x=a\sqrt{2}\frac{\sin(45^\circ+\phi)}{\cos\phi}, \text{ and } y=a\sqrt{2}\frac{\sin(45^\circ-\phi)}{\cos\phi}.$$

Whence, $\text{l}x=\text{l}a+\frac{1}{2}\text{l}2+\text{l}(\sin 45^\circ+\phi)-\text{l}\cos\phi$,
and $\text{l}y=\text{l}a+\frac{1}{2}\text{l}2+\text{l}(\sin 45^\circ-\phi)-\text{l}\cos\phi$;
the angle $\phi$ being determined by the formula $\text{l}\tan\phi=\text{l}b-\text{l}a$.

There are various ways in which the foregoing expressions, and others of the same class, may be adapted to logarithmic computation.

EXAMPLE IX.—Adapt the expression $m\sin A+n\cos A$ to logarithmic computation.

Let $x=m\sin A+n\cos A$; then $x=n\cos A\left\{\frac{m}{n}\tan A+1\right\}$.

Take $\tan^2\phi=\frac{m}{n}\tan A$, whence $1+\frac{m}{n}\tan A=1+\tan^2\phi=\sec^2\phi$.

Therefore, $x=n\cos A\sec^2\phi$, and $\text{l}x=\text{l}n+\text{l}\cos A+2\,\text{l}\sec\phi$,
the angle $\phi$ being obtained from $2\,\text{l}\tan\phi=\text{l}m+\text{l}\tan A-\text{l}n$.

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* In the solution of problems, the answers are not always obtained in a form to which logarithms are immediately applicable, and the formulæ have to undergo certain transformations before the logarithmic tables can be used in calculating the answers. Thus, the values of $m\sin A$ and $n\sin A$ are each calculable by logarithms, but not the value of $m\sin A+n\cos A$. This latter expression has to be *adapted to logarithmic computation*, as it is said, before its value can be found by means of logarithms.

## EXERCISES.

1. Prove the formulæ $\sin^2 A = \frac{\tan^2 A}{1+\tan^2 A}$, $\cos^2 A = \frac{\operatorname{cosec}^2 A - 1}{\operatorname{cosec}^2 A}$

2. Prove the formulæ $\tan A + \cot A = \sec A \operatorname{cosec} A$,
$$\cot^2 A + \cos^2 A = \cot^2 A \cos^2 A.$$

3. Prove geometrically and analytically the formulæ
$$\tan \tfrac{1}{2}A \sin A = 1 - \cos A,\ \cot \tfrac{1}{2}A \sin A = 1 + \cos A.$$

4. From these formulæ it follows that $\frac{1-\cos A}{1+\cos A} = \tan^2 \frac{1}{2}A$.

5. If $\phi$ be less than 45°, $1+\tan\phi = \sqrt{2}\frac{\sin(45°+\phi)}{\cos\phi}$
$$1-\tan\phi = \sqrt{2}\frac{\sin(45°-\phi)}{\cos\phi},\ \text{and}\ \frac{1-\tan\phi}{1+\tan\phi} = \tan(45°-\phi).$$

6. Prove by a geometrical construction that,
$$\cot \tfrac{1}{2}A = \cot A + \operatorname{cosec} A,\ \tan \tfrac{1}{2}A = \operatorname{cosec} A - \cot A.$$

7. Apply Ptolemy's theorem to prove Thomas Simpson's formulæ for multiple angles.
$$\sin(m+1)A = 2 \sin mA \cos A - \sin(m-1)A,$$
$$\cos(m+1)A = 2 \cos mA \cos A - \cos(m-1)A.$$

8. Prove that $\sin 3A = 3 \sin A - 4 \sin^3 A$,
and $\cos 3A = 4 \cos^3 A - 3 \cos A$.

9. Trace the variations in sign of the expressions, $\sin A + \sin 2A$, $\tan A - \sec A$, $\cos A + \operatorname{cosec} 2A$, when the angle A varies from 0° to 180°.

10. Prove $\sin\alpha + \cos\alpha = \cos(\alpha - 45°)\sqrt{2} = \sin(\alpha + 45°)\sqrt{2}$,
and $\sin\alpha - \cos\alpha = \sin(\alpha - 45°)\sqrt{2} = \cos(\alpha + 45°)\sqrt{2}$.

11. Prove that $\sin(A-B) = \sin A \cos B - \sin B \cos A$.
$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$

12. Prove directly, by using Ptolemy's theorem, that
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B},$$
and also deduce analytically these values of $\tan(A \pm B)$.

13. Prove by means of Ptolemy's theorem that, if A be an angle not exceeding two right angles,
$$2 \cos^2 \tfrac{1}{2}A = 1 + \cos A,\ 2 \sin^2 \tfrac{1}{2}A = 1 - \cos A.$$

14. Prove by Ptolemy's theorem, that
$$2 \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B) = \sin A + \sin B.$$
$$2 \sin \tfrac{1}{2}(A-B) \cos \tfrac{1}{2}(A+B) = \sin A - \sin B.$$
$$2 \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B) = \cos A + \cos B.$$
$$2 \sin \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B) = \cos B - \cos A.$$

15. Prove that $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$.
$$\cos(A+B)\cos(A-B) = \cos^2 A - \cos^2 B.$$

16. Find numerical values for the sine, cosine, and tangent, of the angle of 45°.

17. From these expressions calculate values for the sine and cosine of $\frac{1}{2}45°$, $\frac{1}{4}45°$, $\frac{1}{8}45°$.

18. Find numerical values for the sine and cosine of 60°, 30°, 15°

19. Find numerical values for the sine and cosine of 18°, 36°, 54°, 72°.

20. Find expressions for calculating the tangent and secant of 150°, 120°.

21. Find expressions for calculating the cosine and tangent of 108°, 126°.

22. Verify and calculate with 12 decimal places the following expressions:—

$$\sin\frac{180°}{12}=\tfrac{1}{2}\sqrt{(2-\sqrt{3})} \qquad \cos\frac{180°}{12}=\tfrac{1}{2}\sqrt{(2+\sqrt{3})}$$

$$\sin\frac{180°}{24}=\tfrac{1}{2}\sqrt{\{2-\sqrt{(2+\sqrt{3})}\}}, \quad \cos\frac{180°}{24}=\tfrac{1}{2}\sqrt{\{2+\sqrt{(2+\sqrt{3})}\}}$$

$$\sin\frac{180°}{20}=\tfrac{1}{2}\sqrt{\left\{2-\sqrt{\left(\frac{5+\sqrt{5}}{2}\right)}\right\}}, \cos\frac{180°}{20}=\tfrac{1}{2}\sqrt{\left\{2+\sqrt{\left(\frac{5+\sqrt{5}}{2}\right)}\right\}}$$

$$\sin\frac{180°}{40}=\tfrac{1}{2}\sqrt{\left\{2-\sqrt{\left(2+\sqrt{\left(\frac{5+\sqrt{5}}{2}\right)}\right)}\right\}}$$

$$\cos\frac{180°}{40}=\tfrac{1}{2}\sqrt{\left\{2+\sqrt{\left(2+\sqrt{\left(\frac{5+\sqrt{5}}{2}\right)}\right)}\right\}}$$

23. Show that $\sin 18°+\sin 54°=\sin 30°$.

24. When $\sin A=\cdot 5$, what will be the values of $\tan A$ and of $\sec A$

25. What will be the values of $\cos A$ and of $\operatorname{cosec} A$, when
$\tan A=2\cdot 563$?

26. Find geometrically the angles whose tangents are $\frac{1}{\sqrt{3}}$, $\frac{1}{\sqrt{15}}$.

27. What angle has its sine and cotangent equal?

28. Find the angle whose cosine is double of the tangent.

29. Find a value for $\phi$ such that $3\sin\phi=1+2\cos\phi$.

30. Find a value for $\theta$ that will satisfy the two equations,

$$a=a'\sin\theta+b'\cos\theta$$
$$b=a'\cos\theta-b'\sin\theta.$$

31. Find the angle whose cosine is $\frac{\sqrt{3}+\sqrt{2}}{2\sqrt{2}}$.

32. For what value of $\theta$ do the following relations hold?

$$\sin\theta+\cos\theta=0 \qquad \sin\theta-\cos\theta=0$$
$$\sin\theta\cos\theta=1 \qquad \sin\theta+\cos\theta+\tan\theta=\sec\theta.$$

33. Eliminate $\phi$ between the equations

$$m=\operatorname{cosec}\phi-\sin\phi$$
$$n=\sec\phi-\cos\phi.$$

34. Construct the angle whose tangent is $3-\sqrt{2}$.

35. If $\cos\phi=\frac{\cos\theta-e}{1-e\cos\theta}$, prove that $\tan\frac{1}{2}\theta=\tan\frac{1}{2}\phi\sqrt{\frac{1-e}{1+e}}$.

36. Find expressions for the sine and cosine of A, when $\tan A=\frac{a}{b}$.

37. When $m\tan A=n\cot A$, what are the values of $\sin A$ and $\cos A$?

38. Required a value for tan A, when $2\operatorname{cosec} A=3\sec A$.

39. Required a value for tan A, when $2\operatorname{cosec} A=3\sec^2 A$.

40. If $3\cos A+4\sin A=5$, find a value for sin A.

41. Find in function of $\tan\frac{1}{2}A$ a value for the expression

$$\sqrt{\frac{1+\sin A}{1-\sin A}}.$$

42. If $\tan^2\phi=\frac{a}{b}$, express $a+b$ and $a-b$ as functions of $\cos\phi$ and $\cos 2\phi$.

43. Find an expression for sin 2A in function of tan A.

44. If $\tan\frac{A}{2}=2-\sqrt{3}$, find values for sin A and A.

45. If $\tan\theta=\frac{b}{a}$, find a logarithmic expression for computing $a\cos\theta+b\sin\theta$.

46. Find the angle $\theta$ from the expression $\tan 2\theta=8\cos^2\theta-\cot\theta$.

47. If $\theta=\frac{a}{a+b}90^\circ$ and $\phi=\frac{b}{a+b}90^\circ$, then $\sin^2\theta+\sin^2\phi=1$.

48. Find values for A, when $\tan(45^\circ+A)=3\tan(45^\circ-A)$, and when $\tan A+\tan(45^\circ+A)=2$.

49. Prove that $\tan^2\frac{A}{2}$ is always less than tan A sin A.

50. Find values for A, when $\sin\frac{1}{2}A+\cos A=1$,
and when $\sin A+\cos\frac{1}{2}A=1$.

51. Find a value for A, when $\sin A+\cos A=1$.

52. What is the angle whose sine and cosine are together equal to its secant?

53. What are the values of $\theta$, for which $\sin\theta\sqrt{3}=\sqrt{3}-\cos\theta$, and $\cot\theta\tan 2\theta-\tan\theta\cot 2\theta=2$?

54. When $\sin A=\sin B\sin(C+D)$, find a value for tan D.

55. If $r^2\sin^2\theta=\frac{a^2}{b^2}(2br\cos\theta-r^2\cos^2\theta)$, and $a^2-b^2=a^2e^2$, then

$$r=\frac{2b\cos\theta}{1-e^2\sin^2\theta}.$$

56. If $a^2 \tan l' = c^2 \tan l$, then $\tan(l-l') = \frac{(a^2-c^2)\sin l \cos l}{a^2\cos^2 l + c^2 \sin^2 l}$.

57. If $c\cos C = a\cos A + b\cos B$, and $c\sin C = a\sin A + b\sin B$, find the value of $c$ when it is a maximum or a minimum, $a$ and $b$ being fixed quantities.

58. Find values of $x$ and $y$ that will satisfy the equations,

$$a^2\sin\theta'\cos\theta = x^2\sin\theta\cos\theta' + y^2\sin\theta'\cos\theta$$
$$a\cos\theta = x\cos\theta' + y\cos\theta.$$

59. From the expression $a\sin(B-A) = \sin A$, find values for $\sin A$, $\cos A$, and $\tan A$, in function of $a$ and B.

60. If $(1+aa')\tan A = a - a'$, what will be the values for $\sin A$ and $\cos A$?

61. What value of $\alpha$ will satisfy the equation

$$(a-c)\sin\alpha\cos\alpha = \tfrac{1}{2}b(\sin^2\alpha - \cos^2\alpha).$$

62. If $\tan 2\alpha = \frac{b}{c-a}$, then

$$a\cos^2\alpha - b\sin\alpha\cos\alpha + c\sin^2\alpha = \tfrac{1}{2}\{a+c+\sqrt{(b^2+(a-c)^2)}\}.$$
$$a\sin^2\alpha + b\sin\alpha\cos\alpha + c\cos^2\alpha = \tfrac{1}{2}\{a+c-\sqrt{(b^2+(a-c)^2)}\}.$$

63. To prove that, $\quad a'^2 + b'^2 = a^2 + b^2$,
if $a'^2(a^2\sin^2\alpha + b^2\cos^2\alpha) = a^2b^2$, $b'^2(a^2\sin^2\alpha' + b^2\cos^2\alpha') = a^2b^2$,
and $\quad a^2\sin\alpha\sin\alpha' + b^2\cos\alpha\cos\alpha' = 0$.

64. Between the equations $x = a(\cos\theta + \cos 2\theta)$, $y = b(\sin\theta + \sin 2\theta)$, eliminate $\theta$.

65. Determine A when $\sin A + \cos A$ is a minimum; when $\tan A + \cot A$ is a minimum; when $\sec^2 A + \cos^2 A$ is a minimum.

66. Given the sum of two angles, when will the sum of their sines and the sum of their cosines be maxima?

67. Divide an angle into two parts, such that the product of the sines of the parts shall be a maximum.

68. Adapt to logarithmic computation the expression $a+b-c$.

69. Show that if $\cos\phi = \frac{b}{a}$, the expression $\sqrt{(a+b)} + \sqrt{(a-b)}$ becomes $\sqrt{2}\sqrt{(a+b)}\frac{\sin(45^\circ + \frac{1}{2}\phi)}{\cos\frac{1}{2}\phi}$.

70. What values of $x$ will render the expressions $a \pm \sqrt{(a^2 - x^2)}$ calculable by logarithms?

71. Determine $x$, so that $x - \sqrt{(q^2 + x^2)}$ may be adapted to logarithmic computation.

72. Adapt to logarithmic computation the expressions

$$\frac{\sin 2A}{1+\cos 2A}, \quad \frac{\sin 2A}{1-\cos 2A}.$$

73. Adapt to logarithmic computation $\frac{2\sin A+\sin 2A}{2\sin A-\sin 2A}$.

74. When $\cos a=\cos\beta\cos\gamma+\sin\beta\sin\gamma\cos A$, how may the angle $a$ be found by the logarithmic tables?

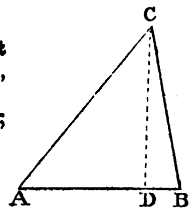
75. If $u=\frac{c}{n}\sin n\theta+c'\cos n\theta$, find $lu$.

---

## LESSON V.

**Relations connecting the sides of a triangle with the angular functions of its angles.**

44. *In a triangle, one of the sides is equal to the sum of the products of the other sides by the cosines of the angles they form with the first side.*



Let ABC be a triangle; then

$$c=a\cos B+b\cos A.$$

From the angle C draw CD at right angles to AB. The triangles ADC, BDC, give

AD=AC cos CAD, DB=BC cos CBD;

Therefore by addition,

$$c=a\cos B+b\cos A.$$

And also,

$$b=c\cos A+a\cos C,$$
$$a=b\cos C+c\cos B.$$

45. *In a triangle, the sines of the angles and the sides opposite to these angles are proportional.*

Let ABC (44) be a triangle, and from the vertex C of one of the angles, let the straight line CD be drawn at right angles to the side AB.

Then (17) $\sin A=\frac{CD}{AC}$, and $\sin B=\frac{CD}{BC}$.

Therefore, $\frac{\sin A}{\sin B}=\frac{CD}{AC}\div\frac{CD}{BC}=\frac{BC}{AC}=\frac{a}{b}$.

Similarly, if a straight line be drawn at right angles to the side AC from the vertex B, or to the side BC from the vertex A, the following relations will be obtained:

$$\frac{\sin A}{\sin C}=\frac{BC}{AB}=\frac{a}{c}, \text{ and } \frac{\sin B}{\sin C}=\frac{AC}{AB}=\frac{b}{c}.$$

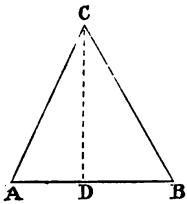
Therefore,

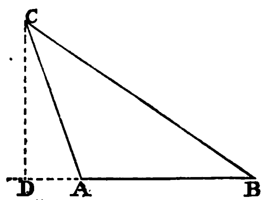
$$\sin A : \sin B : \sin C = a : b : c,$$

which is frequently expressed by writing

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

46. *In a triangle, the cosine of an angle is equal to the quotient obtained by dividing by twice the product of the two sides containing the angle the excess of the sum of the squares of these two sides over the square of the third side.*




Let ABC be the given triangle, and A the angle whose cosine is to be determined. From C let a straight line CD be drawn at right angles to AB, or AB produced, according as the angle A is acute or obtuse.

Then in first figure,

(20) $\cos A = \cos BAC = \frac{AD}{AC}.$

But (Eucl. II., 13),

$BC^2 = AC^2 + AB^2 - 2AB.AD,$

or, $a^2 = b^2 + c^2 - 2c.AD,$

whence, $AD = \frac{b^2+c^2-a^2}{2c},$

therefore, $\frac{AD}{AC} = \frac{b^2+c^2-a^2}{2bc},$

and in second figure,

(20, 29) $\cos A = \cos BAC = -\cos CAD = -\frac{AD}{AC}.$

and (Eucl. II., 12),

$BC^2 = AC^2 + AB^2 + 2AB.AD,$

$a^2 = b^2 + c^2 + 2c.AD,$

$-AD = \frac{b^2+c^2-a^2}{2c},$

$-\frac{AD}{AC} = \frac{b^2+c^2-a^2}{2bc},$

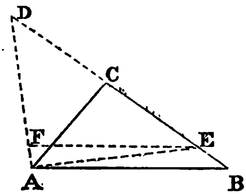
that is to say, whether A be acute or obtuse,

$$\cos A = \frac{b^2+c^2-a^2}{2bc}.$$

In the same way it may be shown that

$$\cos B = \frac{a^2+c^2-b^2}{2ac}, \text{ and } \cos C = \frac{a^2+b^2-c^2}{2ab}.$$

47. *In a triangle, the tangent of half the sum of any two angles is to the tangent of half their difference as the sum of the sides opposite to these angles is to their difference.*



Let ABC be a triangle, and A and B the two angles under consideration. Produce BC, and make CD=CE=CA. Join AD, AE, and through E draw EF parallel to BA.

The angle DEA is equal to half the sum of the angles A and B, and the angle FEA, which is equal to EAB, is equal to half their difference; BD is equal to the sum of the sides BC and CA, and BE is equal to their difference. The angle DAE is a right angle.

Now (18) $\tan \text{DEA} = \frac{\text{AD}}{\text{AE}}$, and $\tan \text{FEA} = \frac{\text{AF}}{\text{AE}}$,

Therefore, $$\frac{\tan \text{DEA}}{\tan \text{FEA}} = \frac{\text{AD}}{\text{AE}} \div \frac{\text{AF}}{\text{AE}} = \frac{\text{AD}}{\text{AF}}.$$

The equiangular triangles ADB, FDE, give (10)

$$\frac{\text{AD}}{\text{DF}} = \frac{\text{BD}}{\text{ED}}, \text{ whence } \frac{\text{AD}}{\text{AF}} = \frac{\text{BD}}{\text{BE}}.$$

Therefore, $$\frac{\tan \text{DEA}}{\tan \text{FEA}} = \frac{\text{BD}}{\text{BE}} \text{ or } \frac{\tan \frac{1}{2}(\text{A}+\text{B})}{\tan \frac{1}{2}(\text{A}-\text{B})} = \frac{a+b}{a-b}.$$

48. *In a triangle, the sine of half one of the angles is to the cosine of half the difference of the other two as the side opposite to the first angle is to the sum of the other sides.*

Let the same construction be made as in (47). The angle ADE is equal to half the angle ACB.

And (28, 29) $$\cos \text{AEF} = \sin \text{AFE} = \sin \text{DAB}.$$

But (45) $$\frac{\sin \text{ADB}}{\sin \text{DAB}} = \frac{\text{AB}}{\text{BD}},$$

Therefore, $$\frac{\sin \frac{1}{2}\text{C}}{\cos \frac{1}{2}(\text{A}-\text{B})} = \frac{c}{a+b}.$$

49. *In a triangle, the cosine of half one of the angles is to the sine of half the difference of the others as the side opposite to the first angle is to the difference of the other sides.*

Let the same construction be made as in (47).

Then, (28, 29) $$\cos \text{ADE} = \sin \text{AED} = \sin \text{AEB},$$

and (45) $$\frac{\sin \text{AEB}}{\sin \text{EAB}} = \frac{\text{AB}}{\text{BE}} \text{ or } \frac{\cos \frac{1}{2}\text{C}}{\sin \frac{1}{2}(\text{A}-\text{B})} = \frac{c}{a-b}.$$

The formula (47) may be obtained by dividing the formula (49) by the formula (48).

# LESSON VI.

**Formulæ and Rules for the solution of Oblique-Angled Triangles—Examples—Exercises.**

## FORMULÆ AND RULES FOR THE SOLUTION OF OBLIQUE-ANGLED TRIANGLES.

50. Four cases present themselves in the solution of oblique-angled triangles, three of them corresponding to the fourth, eighth, and twenty-sixth propositions of the First Book of Euclid's Elements, and the other analogous to the seventh proposition of the Sixth Book.

51. CASE I.—*Given two sides and the contained angle of a triangle, to compute the remaining parts of the triangle.*

Let $a$ and $b$ be the given sides, $a$ being the greater, and C the given angle.

Then $\quad A+B=180°-C$, and $\frac{1}{2}(A+B)=90°-\frac{1}{2}C$.

Also (47)

$$\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\frac{a-b}{a+b}, \text{ whence } \tan\tfrac{1}{2}(A-B)=\frac{a-b}{a+b}\cdot\frac{1}{\tan\frac{1}{2}C};$$

and (48) $$\frac{\sin\frac{1}{2}C}{\cos\frac{1}{2}(A-B)}=\frac{c}{a+b}, \text{ whence } c=(a+b)\frac{\sin\frac{1}{2}C}{\cos\frac{1}{2}(A-B)}$$

$$A=\tfrac{1}{2}(A+B)+\tfrac{1}{2}(A-B), \text{ and } B=\tfrac{1}{2}(A+B)-\tfrac{1}{2}(A-B),$$

or $\quad A=90°-\frac{1}{2}C+\frac{1}{2}(A-B)$, and $B=90°-\frac{1}{2}C-\frac{1}{2}(A-B)$.

The logarithmic formulæ for the computation of $\frac{1}{2}(A-B)$ and of $c$ are obtained as follows:

$$\text{l}\tan\tfrac{1}{2}(A-B)=\text{l}(a-b)-\text{l}(a+b)-\text{l}\tan\tfrac{1}{2}C,$$

$$\text{L}\tan\tfrac{1}{2}(A-B)-10=\text{l}(a-b)-\text{l}(a+b)-[\text{L}\tan\tfrac{1}{2}C-10],$$

or $\text{L}\tan\frac{1}{2}(A-B)=\text{l}(a-b)+10-\text{l}(a+b)+10-\text{L}\tan\frac{1}{2}C$,

or $\quad \text{L}\tan\frac{1}{2}(A-B)=\text{l}(a-b)+c\text{l}(a+b)+c\text{L}\tan\frac{1}{2}C$;*

and $\quad \text{l}c=\text{l}(a+b)+\text{l}\sin\frac{1}{2}C-\text{l}\cos\frac{1}{2}(A-B)$,

$$\text{l}c=\text{l}(a+b)+[\text{L}\sin\tfrac{1}{2}C-10]-[\text{L}\cos\tfrac{1}{2}(A-B)-10],$$

$$\text{l}c=\text{l}(a+b)+\text{L}\sin\tfrac{1}{2}C-\text{L}\cos\tfrac{1}{2}(A-B)$$

or $\quad \text{l}c=\text{l}(a+b)+\text{L}\sin\frac{1}{2}C+c\text{L}\cos\frac{1}{2}(A-B)-10$.

RULE.—To find the half difference of the two angles, add to the logarithm of the difference of the given sides, the arithmetical complements of the logarithm of the sum of the

* The letter $c$ prefixed to l or L signifies that the arithmetical complement of the logarithm is to be taken.

sides, and of the tabular logarithm of the tangent of half the given angle; the sum will be the tabular logarithm of the tangent of half the difference of the two angles.

To find the greater of the angles, to the complement of half the given angle, add the half difference just found; and to find the less of the two angles, subtract the half difference from the complement of half the given angle.

To find the third side, add to the logarithm of the sum of the given sides the tabular logarithm of the sine of half the given angle, and the arithmetical complement of the tabular logarithm of the cosine of half the difference of the other angles; from this result subtract 10; the remainder will be the logarithm of the third side.

By this method the tables will only have to be opened five times, as will appear from the example and exercises.

---

EXAMPLE I.—Two sides, $a$ and $b$, of a triangle measure respectively 1363·9 and 1167 43 feet, and the contained angle 74° 43′ 38″. It is required to compute the remaining side and angles.

| | |
|---|---|
| 1363·9 | 74° 43′ 38″ |
| 1167·43 | |
| 2531·33 sum of sides. | 37 21 49 |
| | 90 |
| 196·47 difference of sides. | 52 38 11 |

*Computation for A and B.*

| | | |
|---|---|---|
| l. 196·47, . . . . | | 2·2932962 |
| cl.2531·33, . . . . | | 6·5966512 |
| cL tan 37° 21′, | 9·8826246 | |
| 49″, | 2141 | 0·1171603 |
| | | 9·0071077 |
| L tan 5° 48′, . | | 9·0067924 |
| 15″ . | | 3153 |

5° 48′ 15″
52 38 11

58 26 26 angle A
46 49 56 angle B
74 43 38 angle C

180

*Computation for c.*

| | | |
|---|---|---|
| l.2531·33, . . . . | | 3·4033488 |
| L sin 37° 21′, . . | | 9·7829614 |
| 49″ . . | | 1352 |
| cL cos 5° 49′, | 9·9977453 | |
| 45″ | 117 | 0·0022430 |
| —10 | | 3·1886884 |

1544·15 feet, the required side.

---

EXERCISES.

1. Given $b=17$ furlongs, $c=12·2584$ furlongs, and A=35° 35′ 4″, to find $a$ and the remaining angles.

2. Given two sides of a triangle equal respectively to 3914·2 and 2288·8 feet, the angle contained between them being 100°, to find the remaining angles and side.

3. Two sides of a triangle, measuring respectively 22842·66 and 23622·21 feet, contain an angle of 44° 19′ 4″. It is required to find the third side and the other angles of the triangle.

4. Given $A=58° 8'$; $b=159$ fathoms, and $c=408$ fathoms, to find B and $a$.

---

52. Case II.—*Given the three sides of a triangle, to compute the angles.*

The formulæ from which the angles may be found are (46)

$$\cos A=\frac{b^2+c^2-a^2}{2bc},\ \cos B=\frac{a^2+c^2-b^2}{2ac},\ \cos C=\frac{a^2+b^2-c^2}{2ab}.$$

These formulæ, however, are not adapted to logarithmic computation, and have to be replaced by others, the values of which can be calculated by means of logarithms.

The relations (39) and (42) of the angular functions of an angle

$$1=\cos^2 \tfrac{1}{2}A+\sin^2 \tfrac{1}{2}A$$
$$\cos A=\cos^2 \tfrac{1}{2}A-\sin^2 \tfrac{1}{2}A$$

give

$$1+\cos A=2\cos^2 \tfrac{1}{2}A, \text{ and } 1-\cos A=2\sin^2 \tfrac{1}{2}A;$$

whence,

$$\tan^2 \tfrac{1}{2}A=\frac{1-\cos A}{1+\cos A}.$$

Substituting for cos A its value (46) in terms of the three sides, this becomes

$$\tan^2 \tfrac{1}{2}A=\frac{1-\dfrac{b^2+c^2-a^2}{2bc}}{1+\dfrac{b^2+c^2-a^2}{2bc}}$$

$$=\frac{2bc-b^2-c^2+a^2}{2bc+b^2+c^2-a^2}$$

$$=\frac{a^2-(b-c)^2}{(b+c)^2-a^2}$$

$$=\frac{(a+b-c)(a+c-b)}{(a+b+c)(b+c-a)}.$$

Let the sum of the three sides be represented by $2s$. Then

$$a+b+c=2s \qquad a+c-b=2s-2b$$
$$a+b-c=2s-2c \qquad b+c-a=2s-2a.$$

The value for $\tan^2 \frac{1}{2}A$ may, therefore, be written,

$$\tan^2 \tfrac{1}{2}A=\frac{(s-b)(s-c)}{s(s-a)},\ \text{whence}\ \tan \tfrac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

The corresponding values for the other angles will be found to be,

$$\tan \tfrac{1}{2}B=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \quad \tan \tfrac{1}{2}C=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

Hence,

$$\mathrm{l} \tan \tfrac{1}{2}A=\tfrac{1}{2}\{\mathrm{l}(s-b)+\mathrm{l}(s-c)-\mathrm{l}s-\mathrm{l}(s-a)\}$$
$$\mathrm{L} \tan \tfrac{1}{2}A-10=\tfrac{1}{2}\{\mathrm{l}(s-b)+\mathrm{l}(s-c)-\mathrm{l}s-\mathrm{l}(s-a)\}$$
$$\mathrm{L} \tan \tfrac{1}{2}A=\tfrac{1}{2}\{\mathrm{l}(s-b)+\mathrm{l}(s-c)-\mathrm{l}s-\mathrm{l}(s-a)+20\};$$

or,

$$\mathrm{L} \tan \tfrac{1}{2}A=\tfrac{1}{2}\{\mathrm{l}(s-b)+\mathrm{l}(s-c)+\mathrm{cl}s+\mathrm{cl}(s-a)\}.$$

Also,

$$\mathrm{L} \tan \tfrac{1}{2}B=\tfrac{1}{2}\{\mathrm{l}(s-a)+\mathrm{l}(s-c)+\mathrm{cl}s+\mathrm{cl}(s-b)\}$$
$$\mathrm{L} \tan \tfrac{1}{2}C=\tfrac{1}{2}\{\mathrm{l}(s-a)+\mathrm{l}(s-b)+\mathrm{cl}s+\mathrm{cl}(s-c)\}.$$

Rule.—To find one of the angles when the three sides are given, add the three sides, take half the sum, and subtract from this half sum each side; add together the logarithms of the differences corresponding to the sides containing the angle, and the arithmetical complements of the logarithms of the half sum, and of the difference corresponding to the side opposite to the angle; take half of this sum. The result will be the tabular logarithm of the tangent of half the angle.

From the expressions

$$2\sin^2 \tfrac{1}{2}A=1-\cos A, \text{ and } 2\cos^2 \tfrac{1}{2}A=1+\cos A,$$

the following values for $\sin \tfrac{1}{2}A$ and $\cos \tfrac{1}{2}A$ can be derived by a process analogous to that used for $\tan \tfrac{1}{2}A$:

$$\sin \tfrac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \cos \tfrac{1}{2}A=\sqrt{\frac{s(s-a)}{bc}}.$$

And from these $\sin A=\dfrac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$.

---

Example II.—The three sides of a triangle are 608·77, 1363·66 and 949·69 feet respectively. It is required to find the three angles.

| | | | |
|---|---|---|---|
| 608·77 | | | |
| 1363·66 | | | |
| 949·69 | | | |
| 2922·12 | | | |
| 1461·06, half sum. | | | |
| 852·29 | | | |
| 97·4 | | | |
| 511·37 | | | |

*Computation for the first angle.*

| | | |
|---|---|---|
| l. | 97·4, . . . | 1·9885590 |
| l. | 511·37, . . . | 2·7087352 |
| cl. | 852·29, . . . | 7·0694126 |
| cl. | 1461·06, . . . | 6·8353320 |
| | | 18·6020388 |
| | | 9·3010194 |
| | L tan 11° 18′, . . | 9·3006383 |
| | 35″ | 3811 |

22° 37′ 10″, the required angle.

*Computation for the second angle.*

| | |
|---|---|
| l. 852·29, . . . . | 2·9305874 |
| l. 511·37, . . . . | 2·7087352 |
| cl. 97·4, . . . . | 8·0114410 |
| cl. 1461·06, . . . . | 6·8353320 |
| | 20·4860956 |
| | 10·2430478 |
| L tan 60° 15′, . | 10·2429480 |
| 20″ | 998 |
| 120° 30′ 40″, the required angle. | |

*Computation for the third angle.*

| | |
|---|---|
| l. 852·29, . . . . | 2·9305874 |
| l. 97·4, . . . . | 1·9885590 |
| cl. 511·37, . . . . | 7·2912648 |
| cl. 1461·06, . . . . | 6·8353320 |
| | 19·0457432 |
| | 9·5228716 |
| L tan 18° 26′, . | 9·5228379 |
| 5″ | 337 |
| 36° 52′ 10″, the required angle. | |

| | | |
|---|---|---|
| 22° | 37′ | 10″ |
| 120 | 30 | 40 |
| 36 | 52 | 10 |
| 180. | | |

## EXERCISES.

5. The three sides of a plane rectilineal triangle are respectively 7694·7, 12584·09, and 5009·49 feet in length; required the angles.

6. Given in a triangle $a=2167$, $b=3921$, $c=4652$, to find the angles.

7. Given $a=9674{\cdot}44$, $b=10156{\cdot}34$, $c=3085{\cdot}88$, to find the angles.

8. Find the angles of a triangle whose three sides measure respectively 9·54, 8·93, and 8·37 chains.

9. In an isosceles triangle the base is 13·0456 chains, and each of the equal sides 78·00028 chains; it is required to find the angles.

10. Given $a=658{\cdot}5$, $b=1028{\cdot}85$, and $c=524\ 97$, to determine the angles.

---

53. CASE III.—*Given a side and two angles of a triangle to find the other sides.*

Let $a$ be the given side, A and B the given angles.

Then $$\mathrm{C}=180°-(\mathrm{A}+\mathrm{B}),$$

and from (45) $$\frac{b}{a}=\frac{\sin \mathrm{B}}{\sin \mathrm{A}},\quad \frac{c}{a}=\frac{\sin \mathrm{C}}{\sin \mathrm{A}},$$

whence $$b=a\,\frac{\sin \mathrm{B}}{\sin \mathrm{A}},\ c=a\,\frac{\sin \mathrm{C}}{\sin \mathrm{A}}.$$

Taking the logarithms

$$\mathrm{l}b=\mathrm{l}a+\mathrm{l}\sin \mathrm{B}-\mathrm{l}\sin \mathrm{A},\ \mathrm{l}c=\mathrm{l}a+\mathrm{l}\sin \mathrm{C}-\mathrm{l}\sin \mathrm{A},$$

$$\left\{\begin{array}{l}\mathrm{l}b=\mathrm{l}a+[\mathrm{L}\sin \mathrm{B}-10]-[\mathrm{L}\sin \mathrm{A}-10],\\ \mathrm{l}c=\mathrm{l}a+[\mathrm{L}\sin \mathrm{C}-10]-[\mathrm{L}\sin \mathrm{A}-10],\end{array}\right.$$

$$\left\{\begin{array}{l}\mathrm{l}b=\mathrm{l}a+\mathrm{L}\sin \mathrm{B}+c\mathrm{L}\sin \mathrm{A}-10,\\ \mathrm{l}c=\mathrm{l}a+\mathrm{L}\sin \mathrm{C}+c\mathrm{L}\sin \mathrm{A}-10.\end{array}\right.$$

RULE.—To find one of the sides, add to the logarithm of the given side the tabular logarithm of the sine of the angle opposite to the required side, and the arithmetical complement of the tabular logarithm of the sine of the angle opposite to the given side; take 10 from the result; the remainder will be the logarithm of the side.

---

EXAMPLE III.—Given one side of a triangle, 2487 yards, the opposite angle, 83° 32′ 25″, and one of the other angles, 47° 43′ 18″, to find the other side.

| | | | |
|---|---|---|---|
| 83° | 32′ | 25″ | |
| 47 | 43 | 18 | |
| 131 | 15 | 43 | |
| 180 | | | |
| 48 | 44 | 17, | the third angle. |

*Computation for the side opposite to the second angle.*

| | | |
|---|---|---|
| l. 2487, . . . . | | 3·3956758 |
| L sin 47° 43′ . . . | | 9·8691301 |
| 18″ . . | | 344 |
| cL sin 83° 32′ | 9·9972280 | |
| 25″ | 60 | |
| | | 0·0027660 |
| —10, | | |
| | | 3·2676063 |

1851·85 yards the required side.

*Computation for the side opposite to the third angle.*

| | |
|---|---|
| l. 2487, . . . . | 3·3956758 |
| L sin 48° 44′ . . . | 9·8760145 |
| 17″ . . | 314 |
| cL sin 83° 32′ 25″ . , | 0·0027660 |
| —10, | |
| | 3·2744877 |

1881·43 yards, the required side.

---

EXERCISES.

11. A side of a triangle measures 24·675 furlongs, the opposite angle, 67° 14′, and the remaining angles 28° 36′ and 84° 10′. Required the sides opposite to these angles.

12. The three angles of a triangle measure respectively 20° 40′ 10″, 69° 21′ 20″, and 89° 57′ 30″, and the side opposite to the last angle is 95·436 chains in length. Required the other sides.

13. Given A=40° 33′ 13″, B=70° 13′, and $b$=153 furlongs, to compute $a$ and $c$.

14. Given $a$=6·254, A=69° 31′ 44″, and B=97° 18′, to compute $b$ and $c$.

---

54. CASE IV.—*Given two sides of a triangle and the angle opposite to one of them, to find the remaining parts.*

Let $a$ and $b$ be the given sides, and A the given angle. The formulæ to be used for the solution are:

$$(45)\quad \sin B = \frac{b}{a}\sin A,\ C = 180° - (A+B),\ \text{and } c = a\,\frac{\sin C}{\sin A}.$$

This case is called the *ambiguous case*, because there are always two values (one less, and the other greater than 90°) for the angle whose sine is equal to the quantity $\frac{b}{a}\sin A$, and, under certain circumstances, both these values will satisfy the conditions of the problem. There will then also be two values for the angle C and the side $c$. Properly speaking, it should be said that this problem sometimes admits of two solutions.

If A be not acute, B must be acute, and therefore the value of B less than 90° is the only one that can be taken.

If A be acute, and $a > b$, then also the smaller of the two values of B is the only one which answers the question.

And further the problem will be impossible if $b \sin A$, the perpendicular from C upon $c$, is greater than $a$.

These observations will be illustrated by the geometrical construction of the problem:—"Given the two sides of a triangle and the angle opposite to one of them, to describe the triangle."

The logarithmic expressions derived from the formulæ are:

$$l\sin B = lb + l\sin A - la,\quad lc = la + l\sin C - l\sin A,$$

$$\begin{cases} L\sin B - 10 = lb + [L\sin A - 10] - la, \\ lc = la + [L\sin C - 10] - [L\sin A - 10, \end{cases}$$

$$\begin{cases} L\sin B = lb + L\sin A + cla - 10, \\ lc = la + L\sin C + cL\sin A - 10. \end{cases}$$

Hence the following

Rule.—To find the angle opposite to the second of the given sides, add to the logarithm of this side the tabular logarithm of the sine of the given angle, and the arithmetical complement of the other given side; subtract 10 from the sum; the remainder will be the tabular logarithm of the sine of the given angle.

To find the third angle, subtract from 180° the sum of the given angle and of the angle just found.

To find the third side apply the rule given in Case III.

---

Example IV.—Given one side of a triangle, 24·67 chains, the opposite angle, 74° 19′, and one of the other sides, 25·3 chains, to compute the third side and the two angles.

*To ascertain whether the problem be possible, and the number of solutions.*

| | | |
|---|---|---|
| l. 25·3, | 1·4031205 | As the side opposite to the given angle is not less than the perpendicular, the problem is possible. But as it is less than the other given side there will be two solutions. |
| L sin 74° 19′, | 9·9835227 | |
| —10, | | |
| | 1·3866432 | |
| l. 24·67, | 1·3921691 | |

| *Computation of the angle opposite to the second side.* | | *Computation of the third angle.* | | | |
|---|---|---|---|---|---|
| | | I. | 74° 19′ | II. | 74° 19′ |
| l. 25.3, . . . . . | 1·4031205 | | 80 53 | | 99 7 |
| cl. 24·67, . . . . | 8·6078309 | | ——— | | ——— |
| L sin 74° 19′, . . . | 9·9835227 | | 155 12 | | 173 26 |
| —10, | ——— | | 180 | | 180 |
| | 9·9944741 | | ——— | | ——— |
| 80° 53′, or 99° 7′ the required angle. | | | 24 48 | | 6 34 |
| | | | the required angle. | | |

*Computation of the third side.*

| *First solution.* | | *Second solution.* | |
|---|---|---|---|
| l. 24·67, . . . . . | 1·3921691 | l. 24·67, . . . . | 1·3921691 |
| L sin 24° 48′, . . . | 9·6226824 | L sin 6° 34′, . . . | 9·0582711 |
| cL sin 74° 19′, . . | 0·0164772 | cL sin 74° 19′, . . | 0·0164772 |
| —10, | ——— | —10, | ——— |
| | 1·0313287 | | 0·4669174 |
| 10·748 chains, the required side. | | 2·9308 chains, the required side. | |

---

EXERCISES.

15. Given $a=1834$ yards, $b=3756$ yards, and $A=17°\ 15'$, to find the other side and the angles.

16. Given $B=54°\ 21'\ 30''$, $b=223{\cdot}54$ yards, and $c=105{\cdot}26$ yards, to compute A, C, and $a$.

17. Given $a=253{\cdot}87$ feet, $b=486{\cdot}54$ feet, and $A=34°\ 12'\ 43''$.

18. The two sides of a triangle measure 14·256 and 16·543 chains respectively and the angle opposite to the former side 21° 14′ 20″. Required the third side, and the angle opposite to it.

---

The following portion of the triangulation carried on by General Roy for the purpose of ascertaining the difference of longitude between the observatories of Greenwich and Paris, is given by Leslie in his Notes on Plane Trigonometry, and will furnish the student with a large variety of exercises in the solution of plane triangles.

The annexed diagram exhibits the combination of triangles extending along the coast of Kent and Sussex, and stretching across the channel.

In the successive triangles contained in the subjoined register, each angle is marked by the single letter affixed to it, and the computed length of its opposite side is given in feet in a line with it.* The asterisk denotes that the angle was deduced from calculation, and not actually observed.

From the train of calculations, it was found that Dover Castle lies 67° 44′ 34″ S.E., and at a distance of 328231 feet, or 62·165 miles, from Greenwich Observatory; and from comparison of this with the results of the operations carried on concurrently by the French astronomers, it was ascertained that the meridian of the observatory of Paris was 2° 19′ 51″ E. from that of Greenwich.

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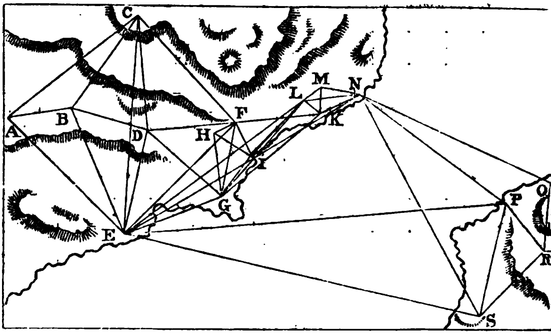
* In several instances, the length of the sides as given by Leslie differs from that inserted in the account of the Trigonometrical Survey published in the 80th volume of the Philosophical Transactions.




| | | | |
|---|---|---|---|
| A | Frant Church. | K | Folkstone Turnpike. |
| B | Goudhurst Church. | L | Paddlesworth. |
| C | Hollingborn Hill. | M | Swingfield Church. |
| D | Tenterden Church. | N | Dover Castle. |
| E | Fairlight Down. | O | Church at Calais. |
| F | Aldington Knoll. | P | Blancnez Signal. |
| G | Lydd Church. | R | Fiennes Signal. |
| H | Ruckinge. | S | Montlambert Signal. |
| I | High Nook. | KL | *The base of verification.* |

**ABC.**

| | | | | |
|---|---|---|---|---|
| A | 27° | 4′ | 36″·13 | 71298·5 |
| B | 136 | 27 | 35·87 | — |
| C | 16 | 27 | 48* | 44391·2 |

**ABE.**

| | | | | |
|---|---|---|---|---|
| A | 43 | 18 | 25·87 | 93629·2 |
| B | 105 | 39 | 28·86 | — |
| E | 31 | 2 | 5·27 | — |

**BCD.**

| | | | | |
|---|---|---|---|---|
| B | 68 | 13 | 19·5 | 71887·5 |
| C | 44 | 38 | 44·04* | 54376·5 |
| D | 67 | 7 | 56·46 | — |

**BDE.**

| | | | | |
|---|---|---|---|---|
| B | 49 | 39 | 35·77 | 71637·2 |
| D | 94 | 59 | 25·81 | 93629·2 |
| E | 35 | 20 | 58·42 | — |

**CDF.**

| | | | | |
|---|---|---|---|---|
| C | 40° | 0′ | 58″·96* | 61777·5 |
| D | 91 | 34 | 22·04 | 96089·8 |
| F | 48 | 24 | 39 | — |

**DFG.**

| | | | | |
|---|---|---|---|---|
| D | 43 | 45 | 28·18 | 47850·9 |
| F | 73 | 0 | 27 | 66169·2 |
| G | 63 | 14 | 9·82* | — |

**DEG.**

| | | | | |
|---|---|---|---|---|
| D | 62 | 32 | 52·51 | 71692·2 |
| E | 54 | 59 | 17·31 | — |
| G | 62 | 27 | 50·18* | 71637·2 |

**EFG.**

| | | | | |
|---|---|---|---|---|
| E | 21 | 18 | 37* | 47850·9 |
| F | 32 | 59 | 23 | — |
| G | 125 | 42 | 0 | 106926·2 |

FGI.

| | | | | |
|---|---|---|---|---|
| F | 33° | 8′ | 46″·1 | 31363·7 |
| G | 24 | 57 | 29·9* | 28185·7 |
| I | 121 | 53 | 44 | — |

FHI.

| | | | | |
|---|---|---|---|---|
| F | 91 | 27 | 19·5 | 28534·5 |
| H | 54 | 19 | 18·5 | — |
| I | 34 | 13 | 22 | 16053 |

FGK.

| | | | | |
|---|---|---|---|---|
| F | 109 | 50 | 39·35 | 84662·8 |
| G | 38 | 2 | 23·76 | 55463·7 |
| K | 32 | 6 | 56·89* | — |

EGL.

| | | | | |
|---|---|---|---|---|
| E | 13 | 38 | 2·95* | 79536·1 |
| G | 154 | 5 | 54·4 | 147392·2 |
| L | 12 | 16 | 2·65 | — |

FIK.

| | | | | |
|---|---|---|---|---|
| F | 76 | 1 | 53·25 | 54708 |
| I | 79 | 41 | 0·5 | — |
| K | 24 | 17 | 6·25 | — |

IKL.

| | | | | |
|---|---|---|---|---|
| I | 14 | 48 | 25·5* | 14714·3 |
| K | 57 | 2 | 0 | 48305·2 |
| L | 108 | 9 | 34·5 | — |

KLM.

| | | | | |
|---|---|---|---|---|
| K | 60 | 27 | 39·5 | 17056·6 |
| L | 70 | 54 | 5·5 | 18525·8 |
| M | 48 | 38 | 15 | — |

KMN.

| | | | | |
|---|---|---|---|---|
| K | 69° | 43′ | 53″·5 | 30560·4 |
| M | 75 | 36 | 40 | 31555·7 |
| N | 34 | 39 | 26·5 | — |

KLN.

| | | | | |
|---|---|---|---|---|
| K | 130 | 11 | 33 | 42562·7 |
| L | 34 | 29 | 42·5 | — |
| N | 15 | 18 | 44·5 | — |

ELN.

| | | | | |
|---|---|---|---|---|
| E | 6 | 6 | 39·48 | — |
| L | 152 | 15 | 25·15 | 186119 |
| N | 21 | 37 | 55·42* | — |

ENP.

| | | | | |
|---|---|---|---|---|
| E | 25 | 33 | 55·02* | 116660 |
| N | 110 | 55 | 29·88* | 252505·6 |
| P | 43 | 30 | 35·15* | — |

ENS.

| | | | | |
|---|---|---|---|---|
| E | 43 | 19 | 58·52 | 168827 |
| N | 87 | 30 | 29·58 | 245786 |
| S | 49 | 9 | 31·9 | — |

NPS.

| | | | | |
|---|---|---|---|---|
| N | 23 | 25 | 0·25 | 77237·2 |
| P | 119 | 41 | 41·64 | — |
| S | 36 | 53 | 18·11 | — |

## EXERCISES.

1. In a right-angled triangle, given one of the angles and the sum of the hypotenuse and of one of the sides, to find the hypotenuse and the sides.

2. Given in a right-angled triangle one of the angles and the difference between the hypotenuse and one of the sides, to find the hypotenuse and the sides.

3. Given one of the angles of a right-angled triangle, and the sum of the sides, to find the sides and the hypotenuse.

4. Given one of the angles of a right-angled triangle, and the difference of the sides, to find the sides and the hypotenuse.

5. Prove that in a right-angled triangle in which $c$ is the hypotenuse, $a \cot \frac{1}{2}A = c + b$.

6. Given a side, the opposite angle, and the sum or difference of the sides containing it, to find these sides and the angles.

7. In any triangle, if a perpendicular be drawn from one of the angles to the opposite side, the products of the segments of this side by the tangents of the adjacent angles are equal.

8. In any triangle, if a straight line be drawn to bisect an angle, and be produced to meet the opposite side, the products of the segments of this side by the sines of the adjacent angles are equal.

9. The straight line that bisects an angle of a triangle divides the opposite side into segments that are proportional to the adjacent sides.

10. Given in a triangle two angles and the sum, or the difference, of the opposite sides, to compute the several parts.

11. Given two sides and the difference of the opposite angles, to compute the several parts of the triangle.

12. If one side of a triangle and the sum of the other two be given, when will the angle opposite to the given side be a maximum?

13. Given the three angles of a triangle and one of the sides, to compute, logarithmically, the segments into which this side is divided by the perpendicular from the opposite angle.

14. The three sides of a triangle being given, to compute, logarithmically, the segments into which one of them is divided by the perpendicular from the opposite angle.

15. Given a side, the difference of the adjacent angles, and the length of the straight line drawn from the middle point of the side to the opposite angle, to compute the several parts of the triangle.

16. The two sides of a triangle are as 3 to 1, and the contained angle is $39° 47' 24''$, find the other angles.

17. In the Trigonometrical Survey of Great Britain, Colonel Mudge found that the logarithms of the numbers expressing the distances in feet between Cheviot and Cross Fell, and between Cheviot and Wisp Hill, were, respectively, 5·4654017 and 5·2672278; and that the angle contained by these lines was $53° 30' 18''$. How may the other angles of the triangle and the distance between Wisp Hill and Cross Fell be determined without finding the numbers corresponding to the given logarithms?

18. In any triangle,

$$(b+c-a)\tan \tfrac{1}{2}A = (a+c-b)\tan \tfrac{1}{2}B = (a+b-c)\tan \tfrac{1}{2}C.$$

19. If in any triangle, $a - b = c \cos \theta$, prove that

$$2\sqrt{ab} \cos \tfrac{1}{2}(A+B) = c \sin \theta, \text{ and } 2\sqrt{ab} \cos \tfrac{1}{2}(A-B) = (a+b) \sin \theta.$$

20. In any triangle,

$$a \sin(B-C) + b \sin(C-A) + c \sin(A-B) = 0.$$

21. Prove that in any triangle,

$$\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1,$$

and conversely.

And that $\sin A+\sin B+\sin C-4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C=0$

$$\sin^2 A+\sin^2 B+\sin^2 C-2\cos A\cos B\cos C-2=0$$

$$\cos\tfrac{1}{2}A+\cos\tfrac{1}{2}B+\cos\tfrac{1}{2}C=4\cos\frac{180^\circ-A}{4}\cos\frac{180^\circ-B}{4}\cos\frac{180^\circ-C}{4}$$

$$\sin\tfrac{1}{2}A+\sin\tfrac{1}{2}B+\sin\tfrac{1}{2}C=1+4\sin\frac{180^\circ-A}{4}\sin\frac{180^\circ-B}{4}\sin\frac{180^\circ-C}{4}$$

$$\sin^2\tfrac{1}{2}A+\sin^2\tfrac{1}{2}B+\sin^2\tfrac{1}{2}C=1-2\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C.$$

22. The sum of the tangents of the angles of a triangle is equal to their product.

23. The sum of the cosines of the halves of any two angles of a triangle is greater than the cosine of the half of the third angle.

24. If one side of a triangle and the opposite angle be given, when will the sum of the other sides be a maximum?

25. If one side of a triangle and the difference of the other sides be given, when will the angle opposite to the given side be a maximum?

26. Given two sides of a triangle and the contained angle, to find the parts into which this angle is divided by the perpendicular from its vertex upon the opposite side.

27. Given two sides of a triangle and the contained angle, to find the parts into which the angle is divided by the straight line drawn from its vertex to the middle point of the opposite side.

28. Given a side, one of the adjacent angles, and the sum, or the difference, of the remaining sides, to compute the several parts of the triangle.

29. If the three sides of a triangle be given, how may the segments be found into which one of the angles is divided by the straight line drawn from its vertex to the middle point of the opposite side?

30. Given two sides of a triangle and the contained angle, to find the segments into which the third side is divided by the bisector of the given angle.

31. Determine the relation between two angles of a triangle, when the squares of their opposite sides are to one another as the adjacent segments of the third side made by the perpendicular from the opposite angle.

32. The distance from the sides of a triangle of the point of intersection of the internal bisectors of the angles, has for its expression

$\sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$, where $2s=a+b+c$.

33. The distance from the sides of a triangle of the point of intersection of the internal bisector of the angle A with the external bisectors of the other angles, has for its value $\sqrt{\dfrac{s(s-b)(s-c)}{s-a}}$.

34. If $r$ be the distance from the sides of a triangle of the point of intersection of the internal bisectors of the angles, and $r'$, $r''$, $r'''$, the distances from the sides of the points in which each of the internal bisectors of the angles A, B, C, meets the external bisectors of the two other angles,

$$\frac{1}{r}=\frac{1}{r'}+\frac{1}{r''}+\frac{1}{r'''}.$$

35. The distance from the angles of a triangle of the point in which the perpendiculars at the middle points of the sides meet, has for its expression $\dfrac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$.

36. If this distance be represented by R,

$$r+4R=r'+r''+r'''.$$

37. Express the sides of a triangle as functions of R and of the angles.

38. Compute the sides of a triangle when the angles and $r$ are given.

39. If the three sides of a triangle be met by a transversal, the product of three alternate segments shall be equal to the product of the other three segments, and conversely.

40. If from a point in the plane of a triangle straight lines be drawn to the angles, and produced to meet the sides, the product of three alternate segments shall be equal to the product of the three remaining segments, and conversely.

41. If a straight line be divided harmonically, the four straight lines joining its extremities and the points of harmonic section with any point not in the line, shall form an harmonic pencil.*

42. If four straight lines form an harmonic pencil, they shall divide harmonically every straight line that cuts them.

43. If $h$, $h'$, $h''$, be the perpendiculars from the angles A, B, C, of a triangle upon the opposite sides, prove that

$$\tan\tfrac{1}{2}A=\sqrt{\frac{(hh'+h'h''-h''h)(h'h''+h''h-hh')}{(hh'+h'h''+h''h)(h''h+hh'-h'h'')}};$$

and find the corresponding expressions for $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$.

44. From the formulæ for the cases of oblique-angled triangles to deduce those used in the solution of right-angled triangles.

---

* Four straight lines issuing from a point are said to form an harmonic pencil when the product of the sines of the angles formed by the two extreme and by the two mean straight lines, is equal to the product of the sines of the angles formed by the first extreme and mean, and by the second extreme and mean.

# LESSON VII.

## Application of Plane Trigonometry to the Measurement of Distances.

55. *To find the distance between two stations, one of which is accessible.*

Let A and B be the two stations whose distance is required, A being accessible.

Assume in a convenient direction a third station C, from which A and B are visible. Measure the distance AC, and the angles CAB, BCA.

Then by (53) $lAB = lAC + L\sin BCA + cL\sin ABC - 10$,
$lBC = lAC + L\sin BAC + cL\sin ABC - 10$.

EXAMPLE I.—Being on the side of a river, and wanting to know the distance to a house that stood on the opposite side, I measured 200 yards in a straight line by the side of the river, and found that at each end of this line, the horizontal angles formed by the other end and the house were 73° 15′ and 68° 2′: what is the distance between each station and the house?

| | |
|---|---|
| 73° 15′ | 180° 0′ |
| 68 2 | 141 17 |
| 141 17 | 38 43 |

| *Distance from first station.* | | *Distance from second station.* | |
|---|---|---|---|
| l. 200, | 2·3010300 | l. 200, | 2·3010300 |
| L sin 68° 2′, | 9·9672679 | L sin 73° 15′, | 9·9811711 |
| cL sin 38° 43′, | 0·2037938 | cL sin 38° 43′, | 0·2037938 |
| —10, | | —10, | |
| | 2·4720917 | | 2·4859951 |
| 296·54 yards. | | 306·19 yards. | |

EXAMPLE II.—To find the perpendicular breadth of a river, I measured 100 yards in a straight line close by one side of it, and at each end of this line I found the angle subtended by the other end and a tree on the opposite bank of the river to be 53° and 79° 12′ respectively.

Let $h$ be the perpendicular distance of the tree B from the measured line, A the first station, and C the second.

Then by (36) $lh = lAB + L\sin BAC - 10$;
and by (53) $lAB = lAC + L\sin BCA + cL\sin ABC - 10$.
Whence $lh = lAC + L\sin BCA + cL\sin ABC + L\sin BAC - 20$.

| | | |
|---|---|---|
| 53° 0′ | l. 100, | 2·0000000 |
| 79 12 | L sin 79° 12′, | 9·9922385 |
| 132 12 | cL sin 47° 48′, | 0·1302962 |
| 180 0 | L sin 53° 0′, | 9·9023486 |
| 47 48 | —20, | |
| | | 2·0248833 |
| | 105·9 yards. | |

56. *To find the distance between two stations not visible the one from the other, but each accessible from a third station.*

Let A and B be the stations whose distance is required, and C the third station.

The distances AC and BC, and the angle ACB, are measured: and the oblique-angled triangle ACB gives (51) (supposing BC>AC)

$$L\tan\tfrac{1}{2}(BAC-ABC)=l(BC-AC)+cl(BC+AC)$$
$$+cL\tan\tfrac{1}{2}ACB,$$
$$lAB=l(BC+AC)+L\sin\tfrac{1}{2}ACB$$
$$+cL\cos\tfrac{1}{2}(BAC-ABC)-10.$$

EXAMPLE III.—Suppose I wanted to know the distance of two places to neither of which there is free access from the other, because of a hill, precipice, or water; and that I measured from each of these places to some convenient third place the distances 735 and 840 links, and found the angle which they subtend at this third place to be 55° 40′. What is the distance between the two places?

840 } Sum, . . 1575
735 } Difference, . 105

2) 55° 40′
27 50

| | | | |
|---|---|---|---|
| l. 105, | 2·0211893 | l. 1575, | 3·1972806 |
| cl. 1575, | 6·8027194 | L sin 27° 50′, | 9·6692250 |
| cL tan 27° 50′, | 0·2773793 | cL cos 7° 12′, | 0·0034381 |
| | | —10, | |
| | 9·1012880 | | 2·8699437 |
| 7° 12′. | | 741·21 links. | |

57. *To find the distance between two objects, neither of which is accessible.*

This problem presents three cases.

I. When both objects can be seen from two known stations.

Let it be required to find the distance between two objects A and B, both of which can be seen from two stations C and D.

The distance CD is measured. The angles DCA, DCB, CDA, CDB, are observed: if the four points A, B, C, D, are in the same plane, the angle BCA is the difference between DCA and DCB; if not, the angle BCA is measured by the instrument. The side AC in the triangle ACD, and the side BC in the triangle BCD are then computed. Lastly, the distance AB is computed in the triangle ABC, in which the sides AC and BC, together with the contained angle BCA, are known.

II. When both objects can be seen from only one of the stations.

Let the objects A and B be both visible from the station C, and B alone visible from the station D.

Select another station E accessible from C and from which A can be seen. Measure the distances CD, CE, and the angles CDB, DCB, BCA, DCA, ECA, and CEA.

The triangles ECA and CDB will give AC and BC respectively; and the triangle ACB will give AB.

III. When both objects cannot be seen from the same station.

Suppose the object A can be seen only from the station C, and the object B only from D.

Take a station E accessible from C and from which A is visible, and a station F accessible from D and from which B can be seen. Measure the distances CE, CD, DF, and the angles CEA, ECA, ACD, CDB, FDB, DFB.

Then AC and BD can be computed in the triangles ACE and BDF; BC and DCB in the triangle DBC; and the required distance AB in the triangle ACB.

EXAMPLE IV.—An observer at station A wishes to know the distance between a house and a mill which are separated from him by a river. He takes another station B, 300 yards from A and towards the same side as the mill; he finds that at A, the angle subtended by B and the mill is 58° 20′, and by the mill and the house 37°; and that at B, the angle subtended by A and the house is 53° 30′, and by the house and the mill 45° 15′.

| | | | |
|---|---|---|---|
| 53° 30′ | 53° 30′ | 53° 30′ | |
| 37 0 | 45 15 | 45 15 | |
| 58 20 | 98 45 | 58 20 | 2)37° 0′ |
| 148 50 | 180 0 | 157 5 | 18 30 |
| 180 0 | 81 15 | 180 0 | |
| 31 10 | | 22 55 | |

| | | | |
|---|---|---|---|
| l. 300, | 2·4771213 | l. 300, | 2·4771213 |
| L sin 53° 30′, | 9·9051787 | L sin 81° 15′, | 9·9949158 |
| cL sin 81° 10′, | 0·2860651 | cL sin 22° 55′, | 0·4096131 |
| —10, | | —10, | |
| 465·98 | 2·6683651 | 761·47 | 2·8816502 |
| 761·47 | | 465·98 | |
| 1227·45 | | 295·49 | |

| | | | |
|---|---|---|---|
| l. 295·49, | 2·4705428 | l. 1227·45, | 3·0890038 |
| cl. 1227·45, | 6·9109962 | L sin 18° 30′, | 9·5014764 |
| cL tan 18° 30′, | 0·4754801 | cL cos 35° 44′, | 0·0905801 |
| | | —10, | |
| | 9·8570191 | | 2·6810602 |
| 35° 44′. | | 479·8 yards, the required distance. | |

58. *Given the angle of depression of an object taken at a known height above the horizontal plane of the object, to find the distance.*

Let A be the object, O the place of observation at a known height OB above the horizontal plane of the object, and COA the angle of depression.

The triangle OBA right-angled in B, and in which OB and OAB are known, will give the distances BA and OA.

EXAMPLE V.—From the top of a ship's mast, 80 feet above the water, the angle of depression of another ship's hull is 20° 18′ 40″. What is the distance between the two ships?

| | |
|---|---|
| l. 80, . . . . . | 1·9030900 |
| L cot 20° 19′, . . . | 10·4315144 |
| 20″, . . | 1291 |
| —10, | |
| | 2·3347335 |

216·14 feet.

59. *To find the distance between two objects in the same horizontal plane by observing their angles of depression from a known height above the plane.*

Two cases present themselves in this problem.

I. When the two objects and the observer's place are in the same vertical plane.

Let A and B be the two objects; O the observer's place, whose distance OC above the horizontal plane of the objects is given; DOA and DOB the given angles of depression.

These angles are respectively equal to the angles OAC and OBC.

The triangles AOB and ACO give (45 and 35)

$$\frac{AB}{AO}=\frac{\sin AOB}{\sin OBA}=\frac{\sin(CBO-CAO)}{\sin CBO},\quad AO=\frac{OC}{\sin OAC};$$

therefore, $$AB=\frac{OC\sin(CBO-CAO)}{\sin CAO.\sin CBO},$$

and $$lAB=lOC+L\sin(CBO-CAO)+cL\sin CAO$$
$$+cL\sin CBO-10.$$

II. When the objects and the observer's place are not in the same vertical plane.

Let O be the observer's place, and A and B the two objects not in the same vertical plane with the place of observation. Besides the angles of depression, the angle subtended at O by A and B is to be measured.

Then OA and OB can be found from the right-angled triangles AOC, BOC; and the required distance AB from the triangle AOB by (51).

$$OA=\frac{OC}{\sin OAC}, \qquad OB=\frac{OC}{\sin OBC}.$$

$$\tan\tfrac{1}{2}(OAB-OBA)=\frac{OB-OA}{OB+OA}\cot\tfrac{1}{2}AOB,$$

$$AB=(OA+OB)\frac{\sin\tfrac{1}{2}AOB}{\cos\tfrac{1}{2}(OAB-OBA)}.$$

Whence,

$$lOA=lOC+cL\sin OAC, \qquad lOB=lOC+cL\sin OBC.$$

$$L\tan\tfrac{1}{2}(OAB-OBA)=l(OB-OA)+cl(OB+OA)+cL\tan\tfrac{1}{2}AOB,$$

$$lAB=l(OA+OB)+L\sin\tfrac{1}{2}AOB+cL\cos\tfrac{1}{2}(OAB-OBA)-10.$$

EXAMPLE VI.—From the top of a lighthouse 196 feet above the level of the sea, the angles of depression of the hulls of two ships are 63° 17′ and 38° 46′ respectively, and the angle they subtend at the lighthouse is 24° 53′. Required the distance of the ships from one another.

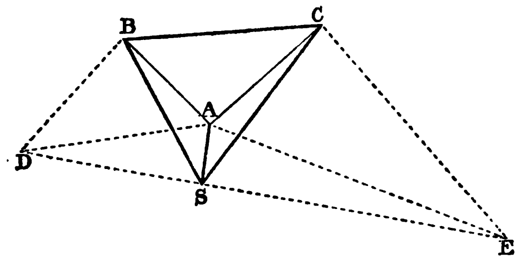
| | | | |
|---|---|---|---|
| l. 196, | 2·2922561 | l. 196, | 2·2922561 |
| $cL$ sin 63° 17′, | 0·0490815 | $cL$ sin 38° 46′, | 0·2033214 |
| 219·42 | 2·3412876 | 313·02 | 2·4955775 |
| 313·02 | | 219·42 | |
| 532·44 | | 93·60 | |
| l. 93·60, | 1·9712758 | l. 532·44, | 2·7262707 |
| cl. 532·44, | 7·2737293 | L sin 12° 26′, | 9·3330511 |
| $cL$ tan 12° 26′, | 0·6566122 | 30″, | 2863 |
| 30″, | 9·9996998 | $cL$ cos 38° 33′, | 0·1067574 |
| —10, | | —10, | |
| | 9·9013471 | | 2·1663655 |
| 38° 33′. | | 146·68 feet required distance. | |

60. *To find the distance of an inaccessible object by taking its angles of depression from two stations at different heights above the plane of the object, the distance between the two stations being known.*

Let A be the object, O and O′ the two stations in the the same vertical direction whose distance from each other is given.

The triangle OAO′, in which the side OO′ and the angles AO′O and AOO′ are known, will give OA or O′A, the distance of the object from either of the stations. The right-angled triangle OCA will then give its distance from the vertical line through the stations.

61. *The distances between three objects on a plane, and the angles which each pair of them subtends at a given station being known, to find the distance of this station from each of the three objects.*




Let A, B, C, be the three objects and S the station. The distances AB, BC, CA, and the angles BSA, ASC, are given: it is required to determine the distances SA, SB, SC.

At the point S draw a straight line at right angles to SA, and through B and C straight lines at right angles to AB and AC, meeting the first perpendicular in D and E. Join AD and AE.

Then because (9) ADB=ASB, and AEC=ASC,

$$AD=\frac{AB}{\sin ASB}, \quad AE=\frac{AC}{\sin ASC}.$$

In the triangle BAC, where the sides are known, the angle A can be computed: but A=ABS+ASB+ACS+ASC.

Therefore,

$$ADE+AED=A-(ASB+ASC),$$

and $$ADE-AED=ABS-ACS.$$

Hence the triangle ADE gives

$$\tan\tfrac{1}{2}(ABS-ACS)=\frac{AE-AD}{AE+AD}\tan\tfrac{1}{2}\{A-(ASB+ASC)\}$$

and the angles ACS and ABS can be found. The distances AS, BS, CS, will then be obtained from the triangles ADS, ABS, and ACS:

$$AS=AD\sin ABS,$$

$$BS=AB\frac{\sin(ABS+ASB)}{\sin ASB},$$

$$CS=AC\frac{\sin(ACS+ASC)}{\sin ASC}.$$

The logarithmic formulæ are as follows:

$$L\sin\tfrac{1}{2}A=\tfrac{1}{2}\{l(s-AC)+l(s-AB)+c l AC+c l AB\}$$

$$lAD=lAB+cL\sin ASB,$$

$$lAE=lAC+cL\sin ASC.$$

$$L\tan\tfrac{1}{2}(ABS-ACS)=l(AE-AD)+cl(AE+AD)+L\tan\tfrac{1}{2}\{A-(ASB+ASC)\}-10,$$

$$lAS=lAD+L\sin ABS-10,$$

$$lBS=lAB+L\sin(ABS+ASB)+cL\sin ASB-10,$$

$$lCS=lAC+L\sin(ACS+ASC)+cL\sin ASC-10.$$

EXAMPLE VII.—Three objects, A, B, C, of which A is the nearest to me, and seen between the others, are at 266·5, 530, and 327·5 yards from each other; and the angle subtended at my station by the two first is 13° 30′, and by the first and last 29° 50′. Required the distances of the three objects from me.

530
266·5
327·5
———
1124
———
562
295·5
234·5

| | |
|---|---|
| l. 295·5, . . . . . | 2·4705575 |
| l. 234·5, . . . . . | 2·3701428 |
| cl. 266·5, . . . . . | 7·5743028 |
| cl. 327·5, . . . . . | 7·4847887 |
| | 19·8997938 |
| 63° 0′, . . . | 9·9498969 |

| | |
|---|---|
| l. 266·5, . . . . . | 2·4256972 |
| cL sin 13° 30′, . . . | 0·6318147 |
| | 3·0575119 |

1141·6
658·32
———
483·28

| | |
|---|---|
| l. 327·5, . . . . | 2·5152113 |
| cL sin 29° 50′, . . . | 0·3032255 |
| | 2·8184368 |

658·32
1141·6
———
1799·92

13° 30′
29 50
———
43 20

2)43° 30′
———
21 40
63 0
———
41 20

| | |
|---|---|
| l. 483·28, . . . . | 2·6841988 |
| cl. 1799·92, . . . . | 6·7447468 |
| L tan 41° 20′, . . . . | 9·9442619 |
| —10 | |
| | 9·3732075 |

13° 17′.

41° 20′ } sum =54° 37′
13 17 } difference=28 3

28° 3′
13 30
———
41 33

54° 37′
29 50
———
84 27

| | |
|---|---|
| l. 1141·6, . . . . | 3·0575119 |
| L sin 28° 3′, . . . . | 9·6723213 |
| —10 | |
| | 2·7298332 |

536·83.

| | |
|---|---|
| l. 266·5, . . . . | 2·4256972 |
| L sin 41° 33′, . . . | 9·8216926 |
| cL sin 13° 30′, . . . | 0·6318147 |
| —10, | |
| | 2·8792045 |

757·19.

| | |
|---|---|
| l. 327·5, . . . . | 2·5152113 |
| L sin 84° 27′, . . . . | 9·9979593 |
| cL sin 29° 50′, . . . | 0·3032255 |
| —10, | |
| | 2·8163961 |

655·23.

# LESSON VIII.

**Application of Plane Trigonometry to the Measurement of Heights—Exercises for computing heights and distances.**

## MEASUREMENT OF HEIGHTS.

62. *To find the height of an object whose base is accessible.*

Two cases have to be considered in this problem.

I. When the base is in the same horizontal plane with the observer.

Let AB be the object. From the base B measure the straight line BC to any convenient point C. Place the instrument at C, and let CF be its height above the horizon. Observe the angle which the line of direction from F to the top of the object makes with the horizontal line FE which meets the line AB in E.

The right-angled triangle ADE, in which the angle ADE and the side DE (equal to CB) are known, will give the side AE: adding the height of the instrument above the horizon, the height of the object AB will be determined.

EXAMPLE I.—Having measured 100 feet from the bottom of a tower, in a direct line from it, on a horizontal plane, I then took the angle of elevation of the top and found it to be 47° 30′; the height of the instrument was 5 feet. Required the height of the tower.

| | |
|---|---|
| l. 100, . . . . . | 2·0000000 |
| L tan 47° 30′, . . . | 10·0379475 |
| —10, | |
| | 2·0379475 |

109·13 feet.

II. When the object is on a plane inclined to the horizon.

Let AB be the object standing on the plane BC inclined to the horizon, whose height it is proposed to compute.

*First method.*—On AB mark the point E, so that BE be equal to the height of the instrument. Measure the distances BC, CD in a direct line from the base B along the inclined plane BD; at the stations C and D, take with the instrument the angles AFE and AGE, subtended by the part of the object between the top and the point E.

The triangle AFG, in which GF is equal to DC, and the angle GAF to AFE − AGE will give AF by (53), and the side AE will be determined from the triangle AFE by (51). Adding the height of the instrument to the value thus found for AE, the whole height AB will become known.

*Second method.*—On AB mark the point E, so that BE be equal to the height of the instrument. Measure the distance BC in a direct line from the base B along the inclined plane; at the station C, take the angle AFE subtended by AE, and the angle of elevation AFH of the top of the object, H being the point in which the horizontal line FH meets the vertical AB.

Then EH and FH will be obtained from the right-angled triangle EFH, and AH from the right-angled triangle AFH. The required height AB, which is equal to AH+EB±EH is therefore determined, the sign + or − to be taken according as the horizontal line FH meets AB, or AB produced.

EXAMPLE II.—I measured a distance of 40 feet from the bottom of an obelisk standing on a declivity, and then took the angle formed by a parallel to the plane, with the straight line drawn to the top 41°; going on in the same direction 60 feet farther, the angle was 23° 45′. What was the height of the obelisk, the height of the instrument being 5 feet from the ground?

| | |
|---|---|
| 41° 0′ | 2)41° 0′ |
| 23 45 | 20 30 |
| 17 15 | |

*From triangle GFA.*

| | |
|---|---|
| l. 60, . . . . . | 1·7781513 |
| L sin 23° 45′, . . . | 9·6050320 |
| cL sin 17° 15′, . . | 0·5279144 |
| —10, | |
| | 1·9110977 |

81·489 } sum =121·489
40 } difference= 41·489

*From triangle AFE.*

| | |
|---|---|
| l. 41·489, . . . . | 1·6179330 |
| cl. 121·489, . . . | 7·9154631 |
| cL tan 20° 30′, . . | 0·4272623 |
| 42° 24′, . . | 9·9606584 |

| | |
|---|---|
| l. 121·489, . . . | 2·0845369 |
| L sin 20° 30′, . . | 9·5443253 |
| cL cos 42° 24, . . | 0·1316758 |
| —10, | |
| | 1·7605380 |

57·615
5·
62·615 feet, the required height.

63. *To find the height of an object when its base is inaccessible.*

This problem presents three cases.

I. When two stations can be taken in the same straight line and in the same horizontal plane with the base of the object.

Let AB be an inaccessible object whose height is required. Take in the same horizontal plane with the base B, and in the same straight line with it, two stations C and D. Measure the distance CD; at the stations C and D, take the the angles AFE and AGE, formed by the straight lines AF

and AG with the horizontal line GF, produced to meet AB in E, the perpendiculars CF and DG representing the height of the instrument.

The triangle AFG gives,

$$\text{l}AF = \text{l}FG + L \sin AGF + cL \sin(AFE - AGF) - 10,$$

and the triangle AFE, right-angled in E, gives,

$$\text{l}AE = \text{l}AF + L \sin AFE - 10.$$

Therefore,

$$\text{l}AE = \text{l}FG + L \sin AGF + cL \sin(AFE - AGF) + L \sin AFE - 20.$$

Then, $AB = AE + EB.$

II. When two stations can be selected in the same straight line, but not in the same horizontal plane with the base.

Let C and D be the two stations. Measure the distance CD; at D take the angle AGE formed by AG, and a parallel GFE to the straight line DCB; and at C take the angles AFH and EFH, formed by the horizontal line FH (meeting the vertical AB in H) with the straight lines AF and FE. The perpendiculars CF and DG represent the height of the instrument.

Then AF will be obtained from the oblique-angled triangle AFG, AH and FH from the right-angled triangle AFH, and EH from the right-angled triangle EFH. And AH+HE+EB, or AH—HE+EB, will be the required height according as the angle of inclination ABC is acute or obtuse.

The logarithmic formulæ for the computation are,

$$\text{l}AF = \text{l}FG + L \sin AGF + cL \sin(AFE - AGF) - 10$$

$$\text{l}AH = \text{l}AF + L \sin AFH - 10, \quad \text{l}FH = \text{l}AF + L \cos AFH - 10$$

$$\text{l}HE = \text{l}FH + L \tan EFH - 10.$$

III. When the two stations cannot be taken in the same straight line with the base.

Let C and D be two stations not in the same straight line with the base B of the object, one of them C being in the same horizontal plane with the base. Let the vertical lines CF and DG represent the height of the instrument. Measure CD; at D take the angle AGF formed by the straight line AG and the parallel GF to the straight line joining the two stations; at C take the angles GFA and AFE, E being the point where the parallel through F to the line joining the station C and the base B meets the vertical AB.

The triangle AGF will give the value of AF, and the triangle AFE will give AE, which added to EB, the height of the instrument gives AB, the required height.

The logarithmic expressions are the same as those used in the first case, with the exception that $c$L sin (AGF+AFG) must be used instead of $c$L sin (AFE—AGF).

If neither of the stations be in the horizontal plane passing through B, a modification similar to that employed in the second case will have to be adopted.

The work will be simplified if the two stations can be taken in the same vertical plane with the top of the object.

Example III.—At a station at the bottom of a hill, the angle of elevation of the top was 56° 12′, and at a station 100 yards from the first on a rising bank opposite to the hill, but in the same vertical plane with the top and the first station, the angle of elevation of the top was found to be 18° 35′, and the angle of depression of the station at the bottom 28° 48′. Required the perpendicular height of the hill, the height of the instrument being 5 feet.

By constructing the diagram for this example, it will be seen that the angle opposite the measured base 100 yards is the difference of the two angles of elevation 56° 12′ and 18° 35′, and the formula to be used will consequently be the same as in the first case of this problem.

$$\begin{array}{r} 56^\circ\ 12' \\ 18\ \ 35 \\ \hline 37\ \ 37 \end{array} \qquad \begin{array}{r} 18^\circ\ 35' \\ 28\ \ 48 \\ \hline 47\ \ 23 \end{array}$$

| | |
|---|---|
| l. 100, | 2·0000000 |
| L sin 47° 23′, | 9·8668189 |
| $c$L sin 37° 37′, | 0·2144028 |
| L sin 56° 12′, | 9·9195929 |
| —20, | |
| | 2·0008146 |

100·19
1·66

101·85 yards, the required height.

64. *To find the height of an object on an inaccessible hill.*

There are two cases in this problem.

I. When two stations can be selected in the same vertical plane with the object.

Let AB be the object; C and D two stations in the same vertical plane with AB; and let the vertical lines CF, DG, represent the height of the instrument. Measure the distance CD; at D take the angles AGF and BGF, and at C the angles AFG and BFG.

Then AG will be found from the triangle AFG, BG from the triangle BFG, and AB from the triangle ABG.

The following are the logarithmic expressions for the computation.

$$\mathrm{l}AG = \mathrm{l}GF + \mathrm{L}\sin AFG + c\mathrm{L}\sin(AGF + AFG) - 10$$

$$\mathrm{l}BG = \mathrm{l}GF + \mathrm{L}\sin BFG + c\mathrm{L}\sin(BGF + BFG) - 10$$

$$\mathrm{L}\tan\tfrac{1}{2}(GBA - GAB) = \mathrm{l}(AG - BG) + c\mathrm{l}(AG + BG) + c\mathrm{L}\tan\tfrac{1}{2}(AGF - BGF)$$

$$\mathrm{l}AB = \mathrm{l}(AG + BG) + \mathrm{L}\sin\tfrac{1}{2}(AGF - BGF) + c\mathrm{L}\cos\tfrac{1}{2}(GBA - GAB) - 10.$$

II. When the two stations are not in the same vertical plane with the object.

In this case the angle AGB is not the difference of the angles AGF and BGF; it is the difference between the angles of elevation of the top and bottom of the object at the station D, and has to be measured, in addition to the measurements taken in the foregoing case. The computation is then conducted as above.

---

### Exercises.

1. From the edge of a ditch, 18 feet wide, surrounding a fort, I took the angle of elevation of the top of the wall, and found it to be 62° 40′. Required the height of the wall, and the length of a ladder necessary to reach from my station to the top of it.

2. From the top of a tower whose height was 120 feet, I took the angle of depression of two trees that stood in a direct line on the same horizontal plane with the bottom of the tower, viz., that of the nearer 57°, and that of the farther 25° 30′. What is the distance between the two trees, and the distance of each from the bottom of the tower?

3. Wanting to know the height of, and my distance from, an object on the other side of a river, which seemed to be on a level with the place where I stood close by the side of the river; and not having room to go backward on the same plane on account of the immediate rise of the bank, I placed a mark where I stood, and measured in a direct line from the object up the hill, whose ascent was so regular as to be considered a straight line, to a distance of 132 yards, where I perceived that I was above the level of the top of the object; I there took the angles of depression of the mark by the river's side 42°, of the bottom of the object 27°, and of its top 19°. Required the height of the object, and the distance of the mark from its bottom.

4. Being on a horizontal plane, and wanting to know the height of an object on the top of an inaccessible hill, I took the angle of elevation of the top of the hill, 40°, and of the top of the object, 51°; measuring then in a direct line from it to the distance of 100 yards farther, I found the angle of the top of the object to be 33° 45′. What is the height of the object?

5. From a window near the bottom of a house which seemed to be on the level with the bottom of a steeple, I took the angle of elevation of the top of the steeple, 47° 15′ 32″, and from another window 18 feet directly above the former, the same angle was 37° 30′ 21″. What then were the height and distance of the steeple?

6. What is the perpendicular height of a cloud whose angles of elevation are 35° 42′ and 64° 18′, taken by two observers at the same time and on the same side of the cloud, at a distance of 880 yards from one another, and so placed that a vertical plane would pass through both their stations and the cloud? Also, what is its distance from the two places of observation?

7. Two ships of war, intending to cannonade a fort, are kept so far from it by the shallowness of the water, that they suspect their guns cannot reach it. In order, therefore, to ascertain the distance, they separate from each other half a-mile, and then each ship observing the angle which the other and the fort subtend, the angles are found to be 85° 15′ and 83° 45′. What is the distance between each ship and the fort?

8. From a ship at sea a point of land was observed to bear east by south, and after sailing north-east 12 miles, it was set again and found to bear south-east by east. How far from the point of land was the last observation made?

9. A May-pole 50 feet 11 inches high, at a certain time will cast a shadow 98 feet 6 inches long. What then is the breadth of a river which, running within 20 feet 6 inches of the foot of a steeple 300 feet 8 inches high, will, at the same time, throw the extremity of its shadow 30 feet 9 inches beyond the stream?

10. Suppose a lighthouse built on the top of a rock: the distance between the place of observation and that part of the rock which is level with the eye and directly under the building is given 310 fathoms; the distance from the top of the rock to the place of observation is 423 fathoms, and from the top of the lighthouse 425. Required the height of the lighthouse.

11. There are two columns standing in a plain, the one 64 feet above it, the other 50. In a right line between them stands a statue, the head of which is 97 feet from the summit of the higher, and 86 feet from that of the lower column, the base of which measures 76 feet to the centre of the figure's base. Required the distance between the tops of the two columns.

12. Suppose the breadth of a well at the top to be 6 feet, and the angle formed by its side and a visual ray from the edge at the top to the opposite side at the bottom to be 18° 30′. What is the depth of the well?

13. At 85 feet distance from the bottom of a tower, the angle of its elevation was found to be 52° 30′ 17″. Required the height of the tower.

14. At a certain station the angle of elevation of an inaccessible tower was 26° 14′ 35″, and, measuring 75 yards in a direct line towards it, the angle of elevation was then found to be 51° 30′ 12″. Required the height of the tower, and the distance from the last station.

15. To find the distance of an inaccessible castle-gate, I measured a line of 73 yards, and at each end of it took the angle of position of the object and the other end, and found the one to be 90°, and the other 61° 45′. Required the distance of the gate from each station.

16. From the top of a tower by the sea-side, 143 feet high, I observed the angle of depression of a ship's hull, then at anchor, to be 35° 27′ 47″. What was its distance from the bottom of the wall?

17. Sailing west-south-west, I saw at some distance a point of land which I set and found its bearing west by north; and after sailing 6 leagues farther, I set it again and found its bearing north-west by west. What was its distance?

18. Observing three steeples, A, B, C, in a town, whose distances asunder are known to be, AB 106½, BC 131, AC 202, fathoms, I took their angles of position from the place D where I stood, which was nearest to the steeple B, and found the angle ADB 13° 30′, and the angle CDB 29° 50′. What was my distance from each of the three steeples?

19. Supposing the station D to be farthest from the steeple B, required to find the distances from it, when AB is 9, BC 6, AC 12, furlongs, and the angles ADB and CDB to be 33° 45′, and 22° 30′ respectively.

20. A ship sailing north-west, two islands appear in sight, of which the one bore north, and the other west north-west: but after sailing 20 leagues, the former bore north-east, and the latter west by south. What is the distance asunder of the two islands?

21. The distance between two stations is 249·56 chains: at the first station two objects subtend with the second station angles of 93° 29′ and 49° 15′; and at the second station the same objects subtend with the first station angles of 57° 36′ and 99° 14′. Required the distance between the two objects.

22. The distances between three points on the map of a country are 335·29, 257·53, and 429·14 chains. It is required to find the distance from each of these points of a fourth point at which the first and second subtend an angle of 23° 36′, and the first and third one of 31° 40′.

# LESSON IX.

Application of Plane Trigonometry to the Measurement of Areas—Triangles and Parallelograms—Quadrilaterals.

## TRIANGLES AND PARALLELOGRAMS.

65. *Given two sides and the contained angle of a triangle, to find its area.*

Let C be the given angle, AC and BC the given sides, and T the area of the triangle.

From A let AD be drawn at right angles to BC.

Then AD=AC sin ACB; and as the area of the triangle has for its expression $\frac{1}{2}$BC.AD, it follows that,

$$T=\tfrac{1}{2}ab\sin C, \text{ whence } lT=la+lb+L\sin C+cl.2-20.$$

The area of a parallelogram, whose sides and angles are known, will therefore be equal to the product of the two adjacent sides by the sine of the contained angle.

66. *Given the three sides of a triangle, to find the area.*

From (65) $$T=\tfrac{1}{2}ab\sin C,$$

and from (52) $$\sin C=\frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}.$$

Therefore, $$T=\sqrt{s(s-a)(s-b)(s-c)},$$

and $$lT=\tfrac{1}{2}\{ls+l(s-a)+l(s-b)+l(s-c)\}.$$

67. *Given two angles and a side in a triangle, to find its area.*

Let A and B be the given angles, and $c$ the given side.

From (65) $T=\frac{1}{2}bc\sin A$, and from (45) $b=c\dfrac{\sin B}{\sin C}$,

Therefore, $$T=\tfrac{1}{2}c^2\frac{\sin A\sin B}{\sin(A+B)},$$

and $lT=2lc+L\sin A+L\sin B+cL\sin(A+B)+cl.2-30.$

68. *Given two sides of a triangle and the angle opposite to one of them, to find the area.*

Let $b$ and $c$ be the two given sides, and C the given angle.

From (54) $L\sin B=lb+clc+L\sin C-10,$

and then by (65), $lT=lb+lc+L\sin(B+C)+cl.2-20.$

69. *The areas of two equiangular triangles are in the duplicate ratio of their homologous sides.* (B. vi., Prop. xix.)

Let T and T′ be the areas of two equiangular triangles, $a$ and $a'$ homologous sides.

Then, $$T=\tfrac{1}{2}a^2\frac{\sin B\sin C}{\sin A},\text{ and }T'=\tfrac{1}{2}a'^2\frac{\sin B\sin C}{\sin A}.$$

Whence, $$\frac{T}{T'}=\frac{a^2}{a'^2}.$$

70. *The areas of equiangular parallelograms have to one another, the ratio which is compounded of the ratios of the sides.* (B. vi., Prop. xxiii.)

Let P and P′ be equiangular parallelograms, $a$ and $b$ two adjacent sides, and A the contained angle in the one; $a'$ and $b'$ two adjacent sides, and A the contained angle in the other.

Then $$P=ab\sin A,\text{ and }P'=a'b'\sin A,$$

which give $$\frac{P}{P'}=\frac{a\,b}{a'b'}=\frac{a}{a'}\cdot\frac{b}{b'}.$$

### QUADRILATERALS.

71. *Given the diagonals of a quadrilateral, and the angle they contain, to find the area.*

Let ABCD be the quadrilateral whose diagonals AC and BD intersecting in O are given, as well as the angle AOB.

The area of the quadrilateral will be equal to the sum of the areas of the four triangles AOB, BOC, COD, DOA, and will have for its expression $\tfrac{1}{2}$AC.BD sin AOB.

72. *Given the four sides of a quadrilateral, and the angle contained by its diagonals, to find the area.*

Let ABCD be the quadrilateral; let $a$, $b$, $c$, and $d$, represent the given sides AB, BC, CD, DA, and O the acute angle formed at the intersection of the diagonals AC and BD, which suppose to be subtended by either of the opposite sides AB or CD.

Then (46)

$$a^2=AO^2+OB^2-2AO.OB\cos O$$
$$b^2=BO^2+OC^2+2BO.OC\cos O$$
$$c^2=CO^2+OD^2-2CO.OD\cos O$$
$$d^2=DO^2+OA^2+2DO.OA\cos O$$

and therefore

$$b^2+d^2-(a^2+c^2)=2\text{AC.BD}\cos\text{O}.$$

But by (71) $\quad \text{Q}=\frac{1}{2}\text{AC.BD}\sin\text{O}.$

Therefore $\quad \text{Q}=\frac{1}{4}\{b^2+d^2-(a^2+c^2)\}\tan\text{O}.$

If $\quad \frac{d}{b}=\tan\phi,\ \frac{c}{a}=\tan\psi,\ \frac{a\sec\psi}{b\sec\phi}=\cos\chi,$

this value will become

$$\text{Q}=\tfrac{1}{4}b^2\sec^2\phi\sec^2\chi\tan\text{O},$$

which is calculable by logarithms.

73. *Given the four sides of a quadrilateral whose opposite angles are together equal to two right angles, to find the angles and the area.*

Let the sides AB, BC, CD, and DA, of the quadrilateral be represented by $a$, $b$, $c$, and $d$ respectively, the diagonal AC by $p$, and the sum $a+b+c+d$ by $2s$.

By (46)

$$p^2=a^2+b^2-2ab\cos\text{B},\ \text{and}\ p^2=c^2+d^2-2cd\cos\text{D},$$

but $\quad \cos\text{D}=-\cos\text{B};$

therefore

$$a^2+b^2-2ab\cos\text{B}=c^2+d^2+2cd\cos\text{B},$$

whence $\quad \cos\text{B}=\dfrac{a^2+b^2-(c^2+d^2)}{2(ab+cd)},$

$$1-\cos\text{B}=1-\frac{a^2+b^2-(c^2+d^2)}{2(ab+cd)}$$

$$=\frac{(c+d)^2-(a-b)^2}{2(ab+cd)}=2\frac{(s-a)(s-b)}{(ab+cd)},$$

$$1+\cos\text{B}=1+\frac{a^2+b^2-(c^2+d^2)}{2(ab+cd)}$$

$$=\frac{(a+b)^2-(c-d)^2}{2(ab+cd)}=2\frac{(s-c)(s-d)}{(ab+cd)};$$

and therefore,

$$\sin\tfrac{1}{2}\text{B}=\sqrt{\frac{(s-a)(s-b)}{ab+cd}},\quad \cos\tfrac{1}{2}\text{B}=\sqrt{\frac{(s-c)(s-d)}{ab+cd}},$$

$$\tan\tfrac{1}{2}\text{B}=\sqrt{\frac{(s-a)(s-b)}{(s-c)(s-d)}},\ \text{Q}=\sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

EXERCISES.

The numerical data of the exercises appended to the several cases of right-angled and oblique-angled triangles, will be sufficient for the computation of the areas of the triangles, and will furnish the student with further practice in the use of the logarithmic tables.

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1. Through a given point in the straight line bisecting a given angle, to draw a straight line such that the triangle it forms with the two sides shall contain a given area.

2. Among all triangles having a common side and equal angles opposite to this side, to determine that whose area is a maximum.

3. If R (Lesson VI, Exercise 36), and the three angles of a triangle be given, the area has for its expression

$$2\,R^2 \sin A \sin B \sin C.$$

4. If the three angles of a triangle be given, the ratio of R to $r$ (Lesson VI, Exercise 34), has for its expression

$$\frac{\sin A+\sin B+\sin C}{2 \sin A \sin B \sin C}.$$

5. Given an angle of a triangle, the straight line joining its vertex with the middle point of the opposite side, and the perpendicular from the angle upon this side, to compute the area, sides, and angles of the triangle.

6. Given in a triangle the area, perimeter, and one of the angles, to compute the sides and angles.

7. Given in a right-angled triangle the hypotenuse and $r$, to find the area, sides, and angles.

8. Given in a right-angled triangle one of the sides and $r$, to compute the area, hypotenuse, and angles.

9. Express the area of a triangle in function of the three sides and R.

10. In a triangle, if T be the area,

$$(1)\ \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C=\frac{T^2}{sabc}$$

$$(2)\ \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C=\frac{sT}{abc}$$

$$(3)\ \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C=\frac{T}{s^2}$$

$$(4)\ T^2=rr'r''r'''.$$

11. Given in a triangle a side, the angle opposite to it, and the area, to determine the remaining parts.

12. Given the area of a triangle and two angles, to calculate the sides.

13. Given the side, the perimeter, and the area of a triangle, to calculate the angles and sides.

14. If the points in which the perpendiculars from the angles of a triangle upon the opposite sides meet these sides, be joined, the triangle thus formed has to the given triangle the same ratio as twice the product of the cosines of the three angles has to 1.

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# LESSON X.

Regular Polygons inscribed and circumscribed, their Areas and Perimeters—Series of polygons formed by continual bisections of the central angles—Limit of the ratio which the successive or the corresponding polygons bear the one to the other—Approximate value of the limit $\pi$.—Area and Circumference of a circle.

## REGULAR POLYGONS INSCRIBED AND CIRCUMSCRIBED, THEIR AREAS AND PERIMETERS.

74. A polygon is said to be regular when all its sides are equal and all its angles equal.

75. The following properties of regular polygons are easily proved:—

(1). If the straight lines bisecting two of the angles of a regular polygon be produced to meet, their point of intersection shall be equally distant from all the vertices of the polygon.

(2). It shall also be equally distant from the points of bisection of the sides.

(3). If at the middle points of two sides of a regular polygon, straight lines be drawn at right angles to the sides and produced to meet, their point of intersection shall be equally distant from the sides, and also from the vertices of the polygon.

(4). The angles at this point (which is called the *centre* of the polygon), subtended by the sides shall be equal to one another.

(5). If the central angles of a regular polygon be bisected, and on the bisecting lines points be taken at the same distance from the centre as the vertices of the polygon, the straight lines joining these points with the vertices of the polygon shall form a regular polygon inscribed in the same circle, but of twice as many sides as the original polygon.

76. The circumference of a circle described from the centre of a regular polygon, with the distance of any of its vertices as radius, is said to *circumscribe* the polygon; and the polygon is said to be *inscribed* in the circle.

The circumference of a circle described from the centre of a regular polygon, with its distance from any of the sides as radius, is said to be *inscribed* in the polygon; and the polygon is said to be *described* about the circle.

77. If through the vertices of a regular polygon straight lines be drawn at right angles to the radii of the circumscribing circle, the polygon formed by the intersections of these straight lines shall be a regular polygon of the same number of sides as the given one, but of greater perimeter and area; and it shall be described about the circle.

The central angle subtended by a side of this polygon shall be equal to the central angle of the original polygon.

78. The perimeter and area of a regular polygon are greater than the circumference and area of the inscribed circle, but less than those of the circumscribing circle.

79. *To find the area and perimeter of a regular polygon, when the side and the number of sides are given.*

Let $a$ be the side and $n$ the number of sides.

Each of the central angles will be equal to $\frac{1}{n}360°$, and the perpendicular distance of the centre from the side will be expressed by $\frac{1}{2}a \cot \frac{1}{2n}360°$. The area of the corresponding triangle will therefore be $\frac{1}{4}a^2 \cot \frac{1}{2n} 360°$.

As the triangles corresponding to each side are equal, and as there are $n$ such triangles in the polygon, the area of the polygon will be $\frac{1}{4}na^2 \cot \frac{1}{2n} 360°$.

The computed values of the expression $\frac{1}{4}n \cot \frac{1}{2n} 360°$ for the several values of $n$ from 3 to 12 are the multipliers given in works on Mensuration in the rule for finding the areas of regular polygons.

The perimeter is $na$.

80. *To find the area and perimeter of a regular polygon, when the number of sides and the radius of the circumscribing circle are given.*

Let $n$ be the number of sides and R the radius of the circumscribing circle.

The central angle will be expressed by $\frac{1}{n}360^\circ$; and the area of the triangle corresponding to each of the sides will be

$$\tfrac{1}{2}R^2 \sin\frac{1}{n}360^\circ, \text{ or } R^2 \sin\frac{1}{2n}360^\circ \cos\frac{1}{2n}360^\circ.$$

The polygon will therefore have for its area

$$nR^2\,\tfrac{1}{2}\sin\frac{1}{n}360^\circ, \text{ or } nR^2 \sin\frac{1}{2n}360^\circ \cos\frac{1}{2n}360^\circ.$$

The perimeter will be expressed by

$$2nR\sin\frac{1}{2n}360^\circ.$$

81. *To find the area and perimeter of a regular polygon, when the number of sides and the radius of the inscribed circle are given.*

Let $n$ be the number of sides, and $r$ the given radius.

Then $\frac{1}{n}360^\circ$ will be one of the central angles, and one of the corresponding triangles will have for its side $2r\tan\frac{1}{2n}360^\circ$, and for its area $r^2\tan\frac{1}{2n}360^\circ$.

The area of the polygon and its perimeter will be

$$nr^2\tan\frac{1}{2n}360^\circ, \text{ and } 2nr\tan\frac{1}{2n}360^\circ.$$

82. *The areas of regular polygons of the same number of sides are to one another as the squares of their sides, or of their radii; and the perimeters are as the sides, or the radii.*

If $n$ be the number of sides, $a$ and $a'$ the sides of two regular polygons, R and R′ their radii, P and P′ their areas, then,

$$(79)\qquad P=\tfrac{1}{4}a^2\cot\frac{1}{2n}360^\circ,\ P'=\tfrac{1}{4}a'^2\cot\frac{1}{2n}360^\circ,$$

whence

$$\frac{P}{P'}=\frac{a^2}{a'^2}.$$

Again,

(80) $P=\frac{1}{2}nR^2 \sin\frac{1}{n}360°,\ P'=\frac{1}{2}nR'^2 \sin\frac{1}{n}360°,$

whence $$\frac{P}{P'}=\frac{R^2}{R'^2}.$$

Also, if $p$ and $p'$ be the perimeters,

(79) $p=na,\ p'=na',$ whence $\frac{p}{p'}=\frac{a}{a'},$

and

(80) $p=2Rn\sin\frac{1}{2n}360°,\quad p'=2R'n\sin\frac{1}{2n}360°,$

whence $$\frac{p}{p'}=\frac{R}{R'}.$$

### SERIES OF POLYGONS FORMED BY CONTINUAL BISECTIONS OF THE CENTRAL ANGLES.

83. *If a regular polygon be inscribed in a given circle, and a series of regular polygons of twice, four times, eight times,...... $2^p$ times, ...... as many sides, be inscribed in the same circle by successive bisections of the central angles, the side of each of these polygons shall be less than that of the polygon which precedes it in the series, but the area and perimeter shall be greater.*

I. Let $s$ and $s'$ be the sides of two consecutive polygons in the series, the first of $n$ sides, and the second of $2n$ sides, and R the radius of the circle.

Then, $s=2R\sin\frac{1}{2n}360°,\qquad s'=2R\sin\frac{1}{4n}360°,$

whence

$$\frac{s}{s'}=\frac{\sin\frac{1}{2n}360°}{\sin\frac{1}{4n}360°}=\frac{\left(2\cos^2\frac{1}{4n}360°\right)}{\cos\frac{1}{4n}360°}=\frac{1+\cos\frac{1}{2n}360°}{\cos\frac{1}{4n}360°}.$$

But the denominator $\cos\frac{1}{4n}360°$ is less than 1; therefore, the ratio of $s$ to $s'$ is one of greater inequality, that is to say, $s'$ is less than $s$.

II. Let P and P′ be the areas of two successive polygons

Then (80) $P=\frac{1}{2}nR^2 \sin\frac{1}{n}360°$, $P'=\frac{1}{2}2nR^2 \sin\frac{1}{2n}360°$,

wherefore, $$\frac{P'}{P}=\frac{2\sin\frac{1}{2n}360°}{\sin\frac{1}{n}360°}=\frac{1}{\cos\frac{1}{2n}360°},$$

hence, $P'>P$.

III. If $p$ and $p'$ represent the perimeters of two successive polygons,

$$p=2Rn\sin\frac{1}{2n}360°, \qquad p'=2R2n\sin\frac{1}{4n}360°,$$

whence, $$\frac{p'}{p}=\frac{2\sin\frac{1}{4n}360°}{\sin\frac{1}{2n}360°}=\frac{1}{\cos\frac{1}{4n}360°},$$

and therefore, $p'>p$.

The polygons, therefore, and their perimeters form increasing series.

84. *If a regular polygon be described about a circle, and a series of regular polygons of twice, four times, eight times,.... $2^p$ times,.........as many sides, be described about the same circle, the side, area, and perimeter of each of these polygons shall be less than the side, area, and perimeter of the polygon that precedes it in the series.*

I. If $s$ and $s'$ be the sides of two consecutive polygons described about the circle; then,

$$s=2R\tan\frac{1}{2n}360°, \qquad s'=2R\tan\frac{1}{4n}360°,$$

and, $$\frac{s'}{s}=\frac{\tan\frac{1}{4n}360°}{\tan\frac{1}{2n}360°}=\tfrac{1}{2}\left(1-\tan^2\frac{1}{4n}360°\right);$$

therefore, $s'<s$.

II. The areas being represented by P and P′,

(81) $P=nR^2\tan\frac{1}{2n}360°$, $P'=2nR^2\tan\frac{1}{4n}360°$;

therefore, $$\frac{P'}{P}=1-\tan^2\frac{1}{4n}360°,$$

or, $P'<P$.

III. As to the perimeters $p$ and $p'$,

(81) $p = 2Rn \tan \frac{1}{2n} 360°$, $\qquad p' = 2R2n \tan \frac{1}{4n} 360°$.

Wherefore, $\qquad \frac{p'}{p} = 1 - \tan^2 \frac{1}{4n} 360°$,

and $p' < p$.

The polygons, therefore, and their perimeters form decreasing series.

## LIMIT OF THE RATIO WHICH THE SUCCESSIVE OR CORRESPONDING POLYGONS BEAR THE ONE TO THE OTHER.

85. *If a regular polygon be inscribed in a given circle, and a regular polygon of the same number of sides described about it, and if by successive bisections of the central angles, a series of regular polygons be inscribed in the circle, and another series described about it, the ratios of the areas of two successive polygons in either series have unity for their limit; and the ratios of the areas of two polygons of the same number of sides in the two series have also unity for their limit.*

Let R be the radius of the given circle; $n$ the number of sides in the primitive polygon in each series; and A its central angle. Let I be the area of the inscribed, and C of the corresponding circumscribed polygon after $p$ bisections of the central angles; I′ and C′ the areas after $p+1$ bisections.

The central angles in the successive polygons of the two series, the areas of the inscribed and those of the circumscribed polygons, will be represented by

$$A \ldots nR^2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A \ldots\ldots\ldots\ldots nR^2 \tan \tfrac{1}{2}A$$

$$\tfrac{1}{2}A \ldots nR^2\, 2 \sin \frac{1}{2^2}A \cos \frac{1}{2^2}A \ldots\ldots\ldots\ldots nR^2\, 2 \tan \frac{1}{2^2}A$$

$$\frac{1}{2^2}A \ldots nR^2\, 2^2 \sin \frac{1}{2^3}A \cos \frac{1}{2^3}A \ldots\ldots\ldots\ldots nR^2\, 2^2 \tan \frac{1}{2^3}A$$

$$\vdots$$

$$\frac{1}{2^p}A \ldots nR^2\, 2^p \sin \frac{1}{2^{p+1}}A \cos \frac{1}{2^{p+1}}A \ldots\ldots nR^2 2^p \tan \frac{1}{2^{p+1}}A$$

$$\frac{1}{2^{p+1}}A \ldots nR^2\, 2^{p+1} \sin \frac{1}{2^{p+2}}A \cos \frac{1}{2^{p+2}}A \ldots nR^2\, 2^{p+1} \tan \frac{1}{2^{p+2}}A$$

$$\vdots$$

Wherefore,

$$\frac{I}{I'}=\frac{1}{2}\,\frac{\sin\frac{1}{2^{p+1}}A\cos\frac{1}{2^{p+1}}A}{\sin\frac{1}{2^{p+2}}A\cos\frac{1}{2^{p+2}}A},\ \text{and}\ \frac{C}{C'}=\frac{1}{2}\,\frac{\tan\frac{1}{2^{p+1}}A}{\tan\frac{1}{2^{p+2}}A}$$

$$=\cos\frac{1}{2^{p+1}}A \qquad\qquad =\frac{1}{1-\tan^2\frac{1}{2^{p+2}}A}.$$

As by carrying on the bisections sufficiently far, $\cos\frac{1}{2^{p+1}}A$ can be made to differ from unity by a quantity less than any assigned one, however small; and as, at the same time, $\tan\frac{1}{2^{p+2}}A$, and consequently $\tan^2\frac{1}{2^{p+2}}A$, will become less than any assigned quantity, it follows that the two ratios have unity for their limit.

Again $$\frac{I}{C}=\frac{\sin\frac{1}{2^{p+1}}A\cos\frac{1}{2^{p+1}}A}{\tan\frac{1}{2^{p+1}}A}=\cos^2\frac{1}{2}A.$$

And as $\cos\frac{1}{2^{p+1}}A$, and consequently $\cos^2\frac{1}{2^{p+1}}A$, can be made to differ from unity by a smaller quantity than any given one, the ratio of the inscribed to the circumscribed polygon of the same number of sides, will have unity for its limit.

It may be proved, by a similar process, that *the ratios of the perimeters of two consecutive polygons in either series have unity for their limit, as have also the ratios of the perimeters of the inscribed and circumscribed polygons of the same number of sides.*

Hence, by constructing a sufficient number of inscribed and corresponding circumscribed regular polygons, it will be always possible to find two of the same number of sides such that the difference of their areas and of their perimeters shall be less than any given quantity, however small.

### APPROXIMATE VALUE OF THE LIMIT $\pi$.

**86.** ***To compute an approximate value for the limit towards which tend the ratios which the areas of the inscribed and circumscribed regular polygons, obtained by successive bisections of the central angles, bear to the square of the radius.***

Let R be the radius; $I, I^{(1)}, I^{(2)}, \ldots\ldots$ and $C, C^{(1)}, C^{(2)}, \ldots\ldots$ the successive polygons inscribed and circumscribed; and A the central angle of I and C.

Then (52) $2\cos\frac{1}{2}A=\sqrt{(2+2\cos A)}$

$$2\cos\frac{1}{2^2}A=\sqrt{[2+\sqrt{(2+2\cos A)}]}$$

$$2\cos\frac{1}{2^3}A=\sqrt{[2+\sqrt{(2+\sqrt{(2+2\cos A)})}]}$$

$$\vdots$$

$$2\cos\frac{1}{2^p}A=\sqrt{(2+2\cos\frac{1}{2^{p-1}}A)}$$

$$=\sqrt{[2+\sqrt{(2+\ldots\ldots+\sqrt{(2+2\cos A)})}]}$$

the number of radical signs corresponding with the index of 2 in the denominator of A on the left hand side.

It follows from this, by (39), that

$$2\sin\frac{1}{2^p}A=\sqrt{(4-4\cos^2\frac{1}{2^p}A)}$$

and thence the value for $\tan\frac{1}{2^p}A$ may be found.

Now $\frac{I}{R^2}=n\sin\frac{1}{2}A\cos\frac{1}{2}A$ and $\frac{C}{R^2}=n\tan\frac{1}{2}A$

$$\frac{I^{(1)}}{R^2}=2n\sin\frac{1}{2^2}A\cos\frac{1}{2^2}A \qquad \frac{C^{(1)}}{R^2}=2n\tan\frac{1}{2^2}A$$

$$\frac{I^{(2)}}{R^2}=2^2n\sin\frac{1}{2^3}A\cos\frac{1}{2^3}A \qquad \frac{C^{(2)}}{R^2}=2^2n\tan\frac{1}{2^3}A$$

$$\vdots \qquad \vdots$$

The values of these several ratios can, therefore, be calculated if the central angle A, and consequently $n$, be given. But as the ratios of $\frac{I}{R^2}$ to $\frac{C}{R^2}$, of $\frac{I^{(1)}}{R^2}$ to $\frac{C^{(1)}}{R^2}$, of $\frac{I^{(2)}}{R^2}$ to $\frac{C^{(2)}}{R^2}$ ...... are the same as the ratios of I to C, of $I^{(1)}$ to $C^{(1)}$, of $I^{(2)}$ to $C^{(2)}$ ......respectively; and as unity is the limit of these latter ratios, it follows that the ratios which the inscribed and corresponding circumscribed polygons bear to the square of the radius tend to equality as the number of times the angle is bisected increases. It will, therefore, be always possible to calculate values for two corresponding ratios that shall differ by as small a quantity as may be desired. The mean between the values for two such ratios will be an approximate value of the limit towards which tend the ratios which

the areas of the inscribed or circumscribed regular polygons bear to the square of the radius, as the number of sides is successively doubled.

Suppose $A=30^\circ$, whence $n=12$, and $\cos A=\frac{1}{2}\sqrt{3}$; then for central angle

$$\frac{360^\circ}{12},\quad 2\cos A=\sqrt{3}=1{\cdot}7320508076$$

$$\frac{360^\circ}{24},\quad 2\cos\tfrac{1}{2}A=\sqrt{(2+1{\cdot}7320508076)}=1{\cdot}9318516525$$

$$\frac{360^\circ}{48},\quad 2\cos\frac{1}{2^2}A=\sqrt{(2+1{\cdot}9318516525)}=1{\cdot}9828897227$$

$$\frac{360^\circ}{96},\quad 2\cos\frac{1}{2^3}A=\sqrt{(2+1{\cdot}9828897227)}=1{\cdot}9957178465$$

$$\frac{360^\circ}{192},\quad 2\cos\frac{1}{2^4}A=\sqrt{(2+1{\cdot}9957178465)}=1{\cdot}9989291743$$

$$\frac{360^\circ}{384},\quad 2\cos\frac{1}{2^5}A=\sqrt{(2+1{\cdot}9989291743)}=1{\cdot}9997322757$$

$$\frac{360^\circ}{768},\quad 2\cos\frac{1}{2^6}A=\sqrt{(2+1{\cdot}9997322757)}=1{\cdot}9999330678$$

$$\frac{360^\circ}{1536},\quad 2\cos\frac{1}{2^7}A=\sqrt{(2+1{\cdot}9999330678)}=1{\cdot}9999832669$$

$$\frac{360^\circ}{3072},\quad 2\cos\frac{1}{2^8}A=\sqrt{(2+1{\cdot}9999832669)}=1{\cdot}9999958167$$

and $\quad 2\sin\frac{1}{2^8}A={\cdot}0040906112,\quad \tan\frac{1}{2^8}A={\cdot}0020453098.$

But (80) $\qquad I=nR^2\sin\frac{1}{2n}360^\circ\cos\frac{1}{2n}360^\circ,$

and (81) $\qquad C=nR^2\tan\frac{1}{2n}360^\circ;$

hence after 7 bisections of the angle $\frac{360^\circ}{12}$,

$$\frac{I^{(7)}}{R^2}=1536\times\sin\frac{1}{2^8}\frac{360^\circ}{12}\times\cos\frac{1}{2^8}\frac{360^\circ}{12},$$

$$=\frac{1536}{4}\times{\cdot}0040906112\times1{\cdot}9999958167$$

$$=3{\cdot}1415828303.$$

$$\frac{C^{(7)}}{R^2}=1536\tan\frac{1}{2^8}\frac{360^\circ}{12},$$

$$=1536\times{\cdot}0020453091$$

$$=3{\cdot}1415958528.$$

These expressions agree to the fourth place of decimals; the mean between them, which is 3·1415893415, will therefore be an approximate value of the limit.

This limit is usually represented by $\pi$. Its value has recently been calculated to 400 decimal places, but by processes much less tedious than the foregoing, which is inserted merely on account of its elementary character, and to give the student some idea of the steps by which an approximate value of the irrational number $\pi$ may be calculated.

The values of the ratios of the perimeters of the polygons to the double of the radius tend to the same limit as those of the ratios of the areas to the square of the radius. Hence $\pi$ is the limit towards which tend the ratios which the perimeters of the inscribed and circumscribed regular polygons bear to the diameter of the circle when the number of sides is successively doubled.

### AREA AND CIRCUMFERENCE OF A CIRCLE.

87. *To find the area and perimeter of a given circle.*

Let R be the radius, and $O$, $o$, be the area and the perimeter of the circle.

If a series of inscribed and circumscribed regular polygons of $n$, $2n$, $2^2n$, ......... sides be formed, the areas and perimeters of the inscribed polygons will be less, and those of the circumscribed polygons will be greater than the circle and its circumference (79). And as at the limit the areas of the inscribed and circumscribed polygons coincide with one another and with the circle, as well as their perimeters and the circumference, and as $\pi$ is the value of the limit towards which tends the ratio which the area has to the square of the radius, or the perimeter to the diameter, it follows that the ratio of the circle to the square of the radius, or of its circumference to the diameter, will be expressed by the same number $\pi$.

Thus $\qquad \dfrac{O}{R^2}=\pi, \qquad \dfrac{o}{2R}=\pi,$

whence $\qquad O=\pi R^2, \qquad o=2\pi R,$

or $\qquad O=\frac{1}{4}\pi D^2, \qquad o=\pi D$, D being the diameter.

88. *Two circles are to one another as the squares of their diameters or radii.*

*The circumferences of two circles are to one another as their diameters or radii.*

# LESSON XI.

Application of Plane Trigonometry to Navigation:—Preliminary Propositions and Definitions—Plane Sailing—Mercator's Sailing—Parallel Sailing—Middle Latitude Sailing.

## PRELIMINARY PROPOSITIONS AND DEFINITIONS.

89. Navigation treats of the methods by which a ship may be conducted from one port to another, or her place determined at any given moment, and of the principles from which the rules and processes furnished for these purposes are derived.

It is the object of this lesson to investigate such of the principles and rules as depend upon the solution of plane right-angled triangles.

The student is however presumed to have made himself thoroughly acquainted with those portions of the Geography Generalized that treat of the form, magnitude, and motions of the earth, of the circles and lines of the terrestrial globe, and of the methods for determining the position of any place by its latitude and longitude.

He will also have to bear in mind the few following propositions and definitions.

(1) *The centres of the equator and of the parallels of latitude are on the polar diameter or axis of the earth.*

(2) *The radii in the equator and in each parallel are at right angles to the axis.*

(3) *The angle of two meridians is measured by the angle formed at the centre of the equator, or of any one of the parallels, by the intersections of these meridians with the equator or the parallel.*

(4) *The measure of an angle at the centre of a circle, and that of the arc intercepted between its sides, are expressed by the same number of degrees.**

(5) *The arcs of circles that subtend equal angles at the centres are proportional to the radii of the circles.*†

(6) *The Latitude of a place*, which is defined to be the arc of a meridian circle intercepted between the equator and the parallel that passes through the place, may also be considered as the *acute angle formed at the centre of the earth by the radius drawn to the place and the intersection of the meridian circle of the place with the equator.*

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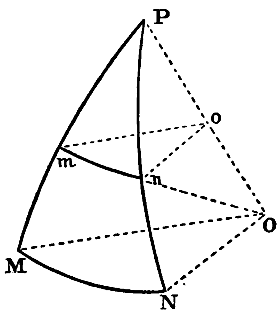
* To prove this, it will be sufficient to take the definition of *Similar Segments*, the 23rd, 24th, 26th, and 27th Propositions of Book III., together with the 5th Definition of Book V., and the first part of the 33rd Proposition of Book VI.

† This property may be derived from the preceding one, taken in connexion with (88).

For this angle and the arc have their measures expressed by the same number of degrees.

90. *The length of an arc of the equator is equal to the length of the corresponding arc of a parallel* (that is, the arc intercepted on the parallel between the meridians passing through the extremities of the equatorial arc) *multiplied by the secant of the latitude of the parallel.*




Let P be the pole; MN an arc of the equator between the meridians PM, PN; *mn* the arc intercepted by them on the parallel of $l$ degrees. Then $MN = mn \sec l^\circ$.

Let O be the centre of the earth, *o* the centre of the parallel; join ON, *on*, O*n*.

The angles MON and *mon* being equal, the arcs MN and *mn* are as the radii ON and *on* (89, 5). But ON being equal to O*n*, and the angle *no*O being a right angle (89, 2), the ratio of ON to *on* is equal to the secant of O*no* (19). Therefore

$$MN = mn \sec Ono = mn \sec l^\circ,$$

because $Ono = nON = l^\circ$ (89, 6).

91. The intersection of the meridian circle of a place, with its sensible horizon, corresponds with the *meridian line* of the place. It is the line indicated by the position of the magnetic needle at the place, allowance being made for what is called the *variation of the compass.*

This meridian line is called the North-and-South line, and one at right angles to it, in the same plane, is the East-and-West line.

Of the four right angles, formed by these lines, the two on the right-hand side of a person, looking towards the north, are called the North-East and the South-East quarters; and the two on the left-hand side, the North-West and the South-West quarters.

Each of the quarters is subdivided into eight equal parts, called points, each of which is further subdivided into fourths.

In the several quarters the angles, or points, are reckoned from the north or the south, towards the east or the west.

A point is equivalent to 11° 15′.

92. The angle, as indicated by the compass, which the intersection of the plane of the sensible horizon and a vertical plane through the fore and aft line of a ship makes with the meridian, is the *compass course* of the ship.

When the compass course has been corrected for variation, deviation, and leeway, the result is called the *true course*.

The course is expressed in points and fourths of a point, or in degrees, minutes, seconds, with the indication of the quarter in which it lies.

93. The straight line in the plane of the horizon, which makes with the meridian an angle equal to the course, is a *rhumb*.

94. The line connecting the several points of the earth through which a ship passes in succession while she sails without altering her course, is called a *rhumb-line*, or *loxodromic curve*.

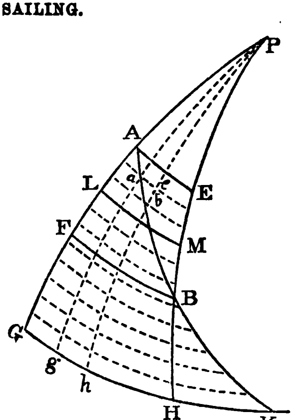
The distance between any two of these points, measured on the rhumb-line, is their *nautical distance*, and is usually expressed in geographical miles.

The rhumb-line is not a circle unless the ship be sailing due north, due south, due east, or due west. For no circle, great or less, makes equal angles with two successive meridian circles, unless it be at right angles to the axis of the earth.

PLANE SAILING.

95. Suppose a ship to have sailed from A to B without altering her course.

Draw the meridians PAG, PBH, through the places of departure and arrival, terminating in the equator GHK. Let ABK be the rhumb-line, produced to meet the equator in K.




If a series of parallels be drawn from the equator to the pole, at a distance of 1′ from each other, the whole arc of the meridians, PG, PH, will be divided into 5,400 equal arcs, each of which will be 1 geographical mile

in length, and may, therefore, without much error, be considered as a straight line. Two of the parallels will pass through A and B, or so near to these places as to warrant for all practical purposes the neglect of the differences, which, for each, will not exceed half a minute, that is, half a mile in length. The number of minutes in the difference between the latitudes from and in, will correspond with the number of parallels from the parallel of A to that of B, and will express in miles the *difference of latitude*.

If the two points, A and B, be on opposite sides of the equator, the number of minutes in the sum of the latitudes from and in, will express in miles the difference of latitude of the two places.

The rhumb-line AB will be divided by the parallels into as many small portions as there are minutes, or miles, in the difference of latitude, and each of these parts may be considered as a straight line. If then meridian circles be conceived to pass through the several points of intersection of the rhumb-line AB with the parallels, a series of triangles (of which $abe$ is the type), will be formed, which will be right-angled in $e$, and may be treated as rectilineal, the side $be$ being a mile in length, and the angle $bae$ being the complement of the course.

Let C° represent the course; $d$, the distance run; $l$ and $L$ the latitude and longitude of departure in minutes; $l'$ and $L'$ the latitude and longitude of arrival in minutes. Let $a'b'e'$ and $a''b''e''$ be the first and last of the series of elementary triangles such as $abe$. Then

$$\begin{array}{lcl} b'e'=a'b'\cos C & \text{and} & a'e'=a'b'\sin C \\ \vdots & & \vdots \\ be=ab\cos C & & ae=ab\sin C \\ \vdots & & \vdots \\ b''e''=a''b''\cos C & & a''e''=a''b''\sin C \end{array}$$

Whence, by addition of equals,

$$b'e'\ldots+be\ldots+b''e''=(a'b'\ldots+ab\ldots+a''b'')\cos C,$$

and $$a'e'\ldots+ae\ldots+a''e''=(a'b'\ldots+ab\ldots+a''b'')\sin C.$$

But $(b'e'\ldots+b'e'\ldots+b''e'')$ is the difference of latitude $(l-l')$; $(a'b'\ldots+ab\ldots+a''b'')$ is the nautical distance $d$. The sum $(a'e'\ldots+ae\ldots+a''e'')$ which is the sum of all the intercepts of the several parallels from A to B, corresponding to the subdivisions of the rhumb-line, is called, in nautical language, the *departure*. Let it be represented by D.

The foregoing expressions may therefore be written,

$$l - l' = d \cos C, \text{ and } D = d \sin C.$$

That is to say, the difference of latitude, the nautical distance, the course, and the departure, are connected by precisely the same relations as are, in a rectilineal right-angled triangle, a side, the hypotenuse, the contained angle, and the remaining side.

All questions in navigation involving three out of these four quantities may, therefore, be solved in the same way as questions concerning the parts of a rectilineal right-angled triangle. Hence, these questions are technically said to belong to *Plane Sailing*.

The same two expressions would have been obtained if the difference of latitude had been conceived to be divided into any parts (whether equal or not) less than 1 mile; or if the nautical distance had been divided into parts, equal or unequal, sufficiently small to be considered as straight lines.

### MERCATOR'S SAILING.

96. The same series of elementary triangles furnish the equations.

$$\begin{array}{lll}
a'e' = b'e' \tan C & \text{whence} & a'e' \sec l = b'e' \tan C \sec l \\
\vdots \quad \vdots & & \vdots \qquad \vdots \\
a e = b e \tan C & & ae \sec (l - n) = b e \tan C \sec (l - n) \\
\vdots \quad \vdots & & \vdots \qquad \vdots \\
a''e'' = b''e'' \tan C & & a''e'' \sec (l' + 1') = b''e'' \tan C \sec (l' + 1'),
\end{array}$$

where $n$ represents the number of minutes between A and the parallel $ae$. Then as (90) $ae \sec (l - n)$ is equal to the intercept $gh$ on the equator between the two meridians that pass through $a$ and $b$, the sum

$$a'e' \sec l \ldots\ldots + ae \sec (l - n) \ldots\ldots + a''e'' \sec (l' + 1')$$

will be equal to GH, the arc of the equator intercepted between the meridian circles PG and PH that pass through the two places A and B, that is, to the difference of longitude $(L - L')$ of these two places.

Therefore, since $b'e' = be = b''e'' = 1$ minute or 1 mile,

$$\begin{aligned}
L - L' = \tan C [&\sec l + \sec (l - 1') + \sec (l - 2') \ldots\ldots\ldots \\
&+ \sec (l - (n - 1')) + \sec (l - n) + \ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots + \sec (l' + 2') + \sec (l' + 1')]
\end{aligned}$$

$L - L'$ being expressed in minutes or in miles.

But $[\sec l + \sec (l-1') + \ldots\ldots + \sec (l'+2') + \sec (l'+1')]$ is the difference between the sum of the secants for every minute from the equator to the latitude $l$, and the sum of the secants for every minute from the equator to the latitude $l'$. Now, the value of the sum of the secants for every minute of the quadrant from the equator upwards has been calculated, and the results have been registered in a table called Table of Meridional Parts, in which the value of the sum of secants corresponding to any given latitude will be found. Thus, corresponding to latitude 20° 19′, stands the number 1245·4, which is the value of

$$(\sec 1' + \sec 2' \ldots\ldots + \sec 1° \ldots\ldots + \sec 20° + \sec 20°\ 1' \ldots + \sec 20°\ 19').$$

If the meridional parts corresponding to the latitude $l$ be represented by M, and those corresponding to the latitude $l'$ by M′, the equation given above may be written,

$$L - L' = (M - M') \tan C,$$

showing that the difference of longitude, the meridional difference of latitude, and the course, are related in the same way as the two sides of a rectilineal right-angled triangle and the angle opposite to the first.

Problems in which the difference of longitude is found in this way belong to *Mercator's Sailing.*

97. From the three formulæ just established

$$l - l' = d \cos C,\ D = d \sin C, \text{ and } L - L' = (M - M') \tan C,$$

it will appear that for a given distance $d$, as the course increases, the difference of latitude diminishes while the departure increases; and that for a given meridional difference of latitude, the difference in longitude increases with the course.

98. If the course is due north or due south, the difference of latitude is equal to the distance, and there is neither departure nor difference of longitude. The formulæ in this case become, since C=0,

$$l - l' = d,\quad D = 0,\quad L - L' = 0.$$

PARALLEL SAILING.

99. When the course is due east or due west, there is no difference of latitude, the departure and distance are equal, and the difference of longitude will be found by multiplying the distance by the secant of the latitude (90.) The two first of the formulæ become, in this case, that is, for C=90°,

$$l - l' = 0,\quad D = d,$$

and the third takes the indeterminate form

$$L - L' = 0 \times \infty$$

The formula giving the true value of the difference of longitude in this case may be obtained as follows:—

Multiply together the first and third formulæ (97),

$$(L-L')(l-l')=(\mathrm{M}-\mathrm{M}')d\sin \mathrm{C},$$

whence,

$$L-L'=\frac{\mathrm{M}-\mathrm{M}'}{l-l'}d\sin \mathrm{C}.$$

But

$\mathrm{M}-\mathrm{M}'=\sec l+\sec(l-1')+\ldots\ldots+\sec(l'+2')+\sec(l'+1')$, and each of these $l-l'$ terms being less than the preceding one, it follows that $\mathrm{M}-\mathrm{M}'$ is greater than $(l-l')\sec(l'+1)$ and less than $(l-l')\sec l$. Therefore, the ratio $\frac{\mathrm{M}-\mathrm{M}'}{l-l'}$, which is always intermediate between $\sec l$ and $\sec l'$, will, at the limit (that is when $l'$ and $l$ become equal, and C becomes 90°) have for its value $\sec l$; and sin C will become equal to 1. Therefore, under these circumstances, the true value of the difference of longitude will be

$$L-L'=d\sec l,$$

which corresponds with the formula (90) already referred to.

Questions solved in this way are said to be worked by *Parallel Sailing.*

### MIDDLE LATITUDE SAILING.

100. The following method, different from that by Mercator's Sailing, is sometimes had recourse to for calculating the difference of longitude.

The departure, which is the sum of the intercepts (95) $a'e',\ldots ae,\ldots a''e''$, on the several parallels of latitude, is greater than the part AE intercepted on the parallel of departure by the meridians PAG and PBH, and less than the part BF intercepted by the same meridians on the parallel of arrival. If the ship were increasing her latitude instead of diminishing it, as supposed in the diagram (95), the reverse would be the case. But whether she increase or decrease her latitude, the departure will have for its measure a number intermediate between the measures of the parts of the parallels from and in intercepted by the meridians of the two places. And as the departure from being 0 at first, gradually attains this value, there will, therefore, be between these two parallels a third parallel such that the portion of it between the two meridians shall have the same measure as the departure, and the difference of longitude shall be the same as if the ship had sailed on this parallel from one meridian to the other.

In practice, the mean of the latitudes from and in is taken as the latitude of this parallel, and the departure D having been computed by the formulæ for plane sailing, the difference of longitude is determined by parallel sailing from the formula $L-L'=\text{D} \sec\left(\frac{l+l'}{2}\right)=d \sin \text{C} \sec\left(\frac{l+l'}{2}\right)$.

Problems solved by this method belong to *Middle Latitude Sailing.*

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EXAMPLE I.—A ship from latitude 47° 30′ N., has sailed S.W. by S. 98 miles. What latitude is she in, and what departure has she made?

She is sailing in the south-west quarter, her course being 3 points, or 33° 45′. Then by (95)

| | |
|---|---|
| l. 98, | 1·9912261 |
| L cos 33° 45′, | 9·9198464 |
| —10, | |
| | 1·9110725 |
| 81·48 | |
| 1° 21′ | |
| 47 30 | |
| 46 9 N, the required latitude in. | |

| | |
|---|---|
| l. 98, | 1·9912261 |
| L sin 33° 45′, | 9·7447390 |
| —10, | |
| | 1·7359651 |
| 54·44 miles, the required departure. | |

EXAMPLE II.—What is the course, and what the distance from Cape Clear (lat. 51° 25′ N., long. 9° 29′ W.), to the island of St. Mary, one of the Azores (lat. 36° 58′ N., long. 25° 12′ W.)

The course is in the south-west quarter. By (96 and 95).

| Long. | Lat. | M. Parts. |
|---|---|---|
| 9° 29′ | 51° 25′ | 3608·7 |
| 25 12 | 36 58 | 2390·2 |
| 15 43=943′ | 14 27=867′ | 1218·5 |

| | |
|---|---|
| l. 943, | 2·9745117 |
| cl. 1218·5, | 6·9141745 |
| | 9·8886862 |
| S. 37° 44′ W, the required course. | |

| | |
|---|---|
| l. 867, | 2·9380191 |
| cL cos 37° 44′, | 0·1048962 |
| | 3·0349153 |
| 1083·7 miles, the required distance. | |

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## EXERCISES.

1. A ship from lat. 40° 18′ S. has sailed N.E. by N. 125 miles. Required her latitude in and the departure made.

2. In sailing 98 miles between the south and east, a ship from lat. 50° 13′ N. has made her departure 82 miles. Required her course, and the latitude in.

3. Yesterday noon we were in lat. 30° 32′ N., and this day at noon we were in lat. 36° 56′ N.; we have run on a direct course between the south and east 5½ knots an hour. Required our course and departure.

4. A ship in lat. 3° 52′ S. is bound to a port bearing N.W. by W. ½ W. in lat. 4° 30′ N. How far does that port lie to the westward, and what is the ship's distance from it?

5. Yesterday noon we were in lat. 33° 15′ N., and bound to a port in lat. 28° 35′ N. lying 196 miles to the west; and this day at noon we were in lat. 30° 20′ N., having made 168 miles of westing. Required the direct course and distance to our intended port.

6. Four days ago we were in lat. 4° 39′ S., long. 83° 16′ E. of Greenwich; and now we are in lat. 0° 35′ N., having made 200 miles of westing. Required our present course and distance to Acheen, in Sumatra (lat. 5° 36′ N., long. 95° 26′ E.)

7. A ship from the lat. 10° 38′ N., and bound to a port lying 282 miles to the S.W. by S., after three successive days of bad weather, finds herself in lat. 3° 58′ N., and 120 miles to the eastward. Required the course and distance to her intended port.

8. Two ships take their departure from the Lizard (lat. 49° 57′ 30″ N.), one bound to St. Michael's, which lies 715 miles to the south and 745 to the west; the other bound to Lisbon, lying 661 miles to the south and 215 to the west, reckoning from the Lizard. They sail in company S.W. ½ W. 610 miles, and then part. What is the direct course and distance of each ship to her port.

9. Sailing between the north and the west, from a port in 1° 30′ south latitude, and then arriving at another port in 1° 34′ north latitude, the ship finds her departure to be 151 miles to the westward of the first. Required the course and distance from port to port.

10. A ship after doubling a cape, and sailing N.E. by N. 45 miles, receives in the night considerable damage from a storm; she then bore directly towards a lighthouse lying 24 miles to the N.W. of the cape, and having run 40 miles, and the day breaking, she discovers a port 42 miles to the N. of the cape. Required her course and distance to that port.

11. What is the course, and what the distance from Arran Light (lat. 53° 8′ N., long. 9° 35′ W.), to Halifax, Nova Scotia (lat. 44° 40′, long. 63° 38′)?

12. A ship takes her departure from Cape Clear (lat. 51° 25′ N., long. 9° 29′ W.), and steers S. 33° 8′ W. till she has run 1024 miles. Required her present latitude and longitude.

13. A ship in latitude 51° 18′ N., long. 22° 6′ W., is bound to a port in the S.E. quarter, distant 1024 miles, and in lat. 37° N. What is her direct course, and how much must she alter her longitude to arrive at the port?

14. A ship from lat. 23° S., sails N.N.E. until her difference of longitude is 7°. Required the latitude in and the distance sailed.

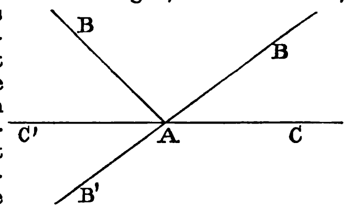
15. A ship in lat. 51° 15′ N., and long. 22° W., sails between south and west until she has made 564 miles of departure and 786 miles of difference of longitude. Required her course, latitude and longitude in, and the distance run.

# LESSON XII.

*Elementary formulæ of Analytical Trigonometry*:—**Angles greater than two right angles, and their angular functions—Negative angles, and their angular functions—Relations connecting the angular functions of an angle with those of an acute angle—Extension of the relations of the angular functions of an angle—Extension of the formulæ for the angular functions of the sum or difference of angles—Important formulæ derived from the foregoing.**

## ANGLES GREATER THAN TWO RIGHT ANGLES, AND THEIR ANGULAR FUNCTIONS.

101. The angles hitherto considered have not exceeded two right angles, or 180°. These angles, it has been seen, may be conceived as formed by the various inclinations to the straight line AC, which the straight line AB has in revolving from the initial position AC until it coincides with the continuation AC′ of the straight line AC. In like manner, if the revolving line after passing through the position AC′ continue its motion towards AC, the inclination which it has to the initial line AC in any of its new positions, AB′ for instance, is to be regarded as an angle; and thus arises the consideration of an angle greater than two or than three right angles. If the line AB, after returning to the position AC, continue its revolutions, angles greater than 4, 5, 6,......*n* right angles, will be formed.




It is true that for all angles exceeding four right angles, or 360°, the revolving line returns to some one of the positions it previously occupied in passing round from the initial position, and if A be the angle not greater than four right angles formed by the revolving and initial lines, the angles formed by them, when the revolving line resumes the same position after 1, 2, 3......*n* revolutions will be expressed by

$$360°+A, \quad 2.360°+A, \quad 3.360°+A, \ldots\ldots n.360°+A.$$

Thus, when the revolving line makes with the initial line an angle of 2012°, it is in the same position with reference to this initial line as it occupied when it made an angle of

$$212°, \quad 572°, \quad 932°, \quad 1292°, \text{ and } 1652°.$$

102. The angular functions depending only upon the relative positions of the revolving and initial lines, will be the same for the angles $360°+A$, $2.360°+A, \ldots\ldots n.360°+A$, as for the angle A.

The following relations, wherein A expresses an angle not exceeding four right angles, will, therefore, hold:

$$\sin(n.360°+A)=\sin A, \quad \cos(n.360°+A)=\cos A$$
$$\tan(n.360°+A)=\tan A, \quad \cot(n.360°+A)=\cot A$$
$$\sec(n.360°+A)=\sec A, \ \text{cosec}\,(n.360°+A)=\text{cosec}\,A.$$

NEGATIVE ANGLES, AND THEIR ANGULAR FUNCTIONS.

103. The straight line AB′ (101) makes with the initial line AC two angles; one formed by supposing the line to revolve, as heretofore, from right to left of a spectator placed at the vertex A, and looking along the initial line towards C; the other, by supposing the line to revolve in the opposite direction, that is, from left to right of the same spectator. In order to distinguish the one from the other the two angles which the revolving line makes with the initial line, it has been agreed to affect them with opposite signs, and to prefix the sign +, or no sign, to the angles of the former class, and the sign — to those of the latter.

Thus the angles

| | | | | | |
|---|---|---|---|---|---|
| | 212° | 572° | 932° | 1292° | 1652° |
| and | —148° | —408° | —768° | —1128° | —1488°, |

are all formed by the same relative positions of the revolving and initial lines, but the direction in which the revolutions take place is different. For the first series the line is supposed to move from right to left, and for the second series from left to right.

104. The foregoing observations show that

$$\left\{\begin{array}{l}\sin(-n.360°-A)=\sin(-A),\\ \tan(-n.360°-A)=\tan(-A),\\ \sec(-n.360°-A)=\sec(-A),\end{array}\right.$$

$$\left\{\begin{array}{l}\cos(-n.360°-A)=\cos(-A),\\ \cot(-n.360°-A)=\cot(-A),\\ \text{cosec}\,(-n.360°-A)=\text{cosec}\,(-A),\end{array}\right.$$

and that

$$\sin(A-n.360°)=\sin A, \quad \cos(A-n.360°)=\cos A,$$
$$\tan(A-n.360°)=\tan A, \quad \cot(A-n.360°)=\cot A,$$
$$\sec(A-n.360°)=\sec A, \quad \text{cosec}\,(A-n.360°)=\text{cosec}\,A.$$

105. *If two angles be equal in magnitude but have opposite signs, the absolute values of their angular functions are equal, and their sines, tangents, cotangents, and cosecants have opposite signs, while their secants and cosines have the same sign.*

For, in the first place, the angles being equal in size the ratios which constitute their angular functions will be respectively equal.

In the second place, in the ratios for the sines the antecedents only are measured in opposite directions; and the same will be the case with the ratios for the tangents.

In the ratios for the cotangents and cosecants, the consequents alone are measured in opposite directions. But both for the secants and for the cosines, the terms of the respective ratios are measured in the same direction.

These relations are expressed by writing

$$\sin(-A) = -\sin A, \quad \cos(-A) = \cos A,$$
$$\tan(-A) = -\tan A, \quad \cot(-A) = -\cot A,$$
$$\sec(-A) = \sec A, \quad \operatorname{cosec}(-A) = -\operatorname{cosec} A.$$

106. If the position of the revolving line and the direction in which it moves be carefully attended to in each case, there will be no difficulty in understanding the following relations:

$$\left\{\begin{array}{ll} \sin(n.360°\pm A) = \pm\sin A & \cos(n.360°\pm A) = \cos A \\ \tan(n.360°\pm A) = \pm\tan A & \cot(n.360°\pm A) = \pm\cot A \\ \sec(n.360°\pm A) = \sec A & \operatorname{cosec}(n.360°\pm A) = \pm\operatorname{cosec} A \end{array}\right.$$

$$\left\{\begin{array}{ll} \sin(A\pm n.360°) = \sin A & \cos(A\pm n.360°) = \cos A \\ \tan(A\pm n.360°) = \tan A & \cot(A\pm n.360°) = \cot A \\ \sec(A\pm n.360°) = \sec A & \operatorname{cosec}(A\pm n.360°) = \operatorname{cosec} A \end{array}\right.$$

### RELATIONS CONNECTING THE ANGULAR FUNCTIONS OF AN ANGLE WITH THOSE OF AN ACUTE ANGLE.

107. *The angular functions of any given angle greater than a right angle are equal in absolute value to the corresponding angular functions of an angle less than one right angle.*

Any positive angle not greater than four right angles may be represented by

$$\alpha, \quad 180°-\alpha, \quad 180°+\alpha, \quad \text{or } 360°-\alpha,$$

where $\alpha$ is a positive angle less than $90°$; and any angle, positive or negative, how great soever its measure, will be obtained from one of the expressions

$$\pm n.360°\pm\alpha, \quad \text{or } \pm n.360°+180°\pm\alpha,$$

by giving to $n$ a suitable value from the series of natural numbers 0, 1, 2, 3, 4,..............

$$\text{But}\ \begin{cases} \sin(\pm n.360^\circ \pm a) & = \sin(\pm a) \quad = \pm \sin a \\ \sin(\pm n.360^\circ + 180^\circ \pm a) & = \sin(180^\circ \pm a) = \mp \sin a \end{cases}$$

$$\begin{cases} \tan(\pm n.360^\circ \pm a) & = \tan(\pm a) \quad = \pm \tan a \\ \tan(\pm n.360^\circ + 180^\circ \pm a) & = \tan(180^\circ \pm a) = \pm \tan a \end{cases}$$

$$\begin{cases} \cos(\pm n.360^\circ \pm a) & = \cos(\pm a) \quad = \cos a \\ \cos(\pm n.360^\circ + 180^\circ \pm a) & = \cos(180^\circ \pm a) = -\cos a. \end{cases}$$

Hence the sine, the tangent, and the cosine of any angle are equal in absolute value to the sine, the tangent, and the cosine of an angle less than one right angle.

In a similar way, the property may be proved for the secant, cotangent, and cosecant.

108. *To find an angle less than* 90° *whose angular functions are equal in absolute value to those of a given angle.*

Find a multiple of 360° such that the difference between it and the given angle shall be less than 180°. If this difference be less than 90°, the problem is solved; if not, the supplement of this difference will be the required angle.

If 1659° be the given angle, the difference between it and five times 360° will be found to be 147°, and then 180°—147°, or 33° will be the angle less than 90°, whose angular functions will have the same absolute values as those of the given angle 1659°.

109. *To find all the angles that have the same given angular function with a given angle.*

I. Let $\frac{p}{q}$ be the sine of the given angle A, and let $a$ represent the positive angle less than 90°, whose sine is $\frac{p}{q}$.

If $\frac{p}{q}$ be positive, the formulæ (107)

$$\pm n.360^\circ + a, \text{ and } \pm n.360^\circ + 180^\circ - a,$$

will contain all the angles whose sines are equal to $\frac{p}{q}$.

If $\frac{p}{q}$ be negative, the angles will be contained in the formulæ (107)

$$\pm n.360^\circ - a, \text{ and } \pm n.360^\circ + 180^\circ + a.$$

II. Let $\frac{p}{q}$ be the tangent of the given angle A, and $a$ the positive angle less than 90°, whose tangent is equal to $\frac{p}{q}$.

The formulæ

$$\pm n.360°+a, \text{ and } \pm n.360°+180°+a,$$

will give the required angle when $\frac{p}{q}$ is positive, and the formulæ $\pm n.360°-a$, and $\pm n.360°+180°-a$,

when $\frac{p}{q}$ is negative.

III. Let $\frac{p}{q}$ be the cosine of the given angle A, and $a$ the angle less than 90°, whose cosine is $\frac{p}{q}$.

The angles will be obtained from the formulæ

$$\begin{cases} \pm n.360°\pm a, & \text{if } \frac{p}{q} \text{ be positive,} \\ \pm n.360°+180°\pm a, & \text{if } \frac{p}{q} \text{ be negative.} \end{cases}$$

EXTENSION OF THE RELATIONS OF THE ANGULAR FUNCTIONS OF AN ANGLE.

110. The complement of a positive angle greater than 90° is negative.

The complement of a negative angle is positive.

The supplement of a positive angle greater than 180° is negative.

The supplement of a negative angle is positive.

111. *The complementary and supplementary relations given* (28) *and* (29) *hold for all angles, as also the relations proved in Lesson IV.*

The extension of the relations in question to all angles will offer no difficulty after the foregoing details. For instance, to show that the relation $\cos A = \cos^2 \frac{1}{2}A - \sin^2 \frac{1}{2}A$, proved (42) when the angle A does not exceed two right-angles, is true for all angles, the following process may be adopted.

The angle A, whatever be its magnitude, will be expressed by one of the formulæ,

$$\pm n.360°\pm a, \text{ or } \pm n.360°+180\pm a,$$

in which $a$ is an angle less than a right-angle, and for which, consequently, by (42),

$$\cos a = \cos^2 \tfrac{1}{2}a - \sin^2 \tfrac{1}{2}a.$$

I. If the angle A belong to the first expression then,

$$\cos A = \cos(\pm n.360^\circ \pm a) = \cos a.$$

$$\cos \tfrac{1}{2}A = \cos(\pm n.180^\circ \pm \tfrac{1}{2}a)$$
$$= \begin{cases} \cos(\pm \tfrac{1}{2}a) = \cos \tfrac{1}{2}a, \text{ if } n \text{ be even;} \\ \cos(\pm 180^\circ \pm \tfrac{1}{2}a) = -\cos \tfrac{1}{2}a, \text{ if } n \text{ be odd;} \end{cases}$$
$$\sin \tfrac{1}{2} A = \sin(\pm n.180^\circ \pm \tfrac{1}{2}a)$$
$$= \begin{cases} \sin(\pm \tfrac{1}{2}a) = \pm \sin \tfrac{1}{2}a, \text{ if } n \text{ be even,} \\ \sin(\pm 180^\circ \pm \tfrac{1}{2}a) = \mp \sin \tfrac{1}{2}a, \text{ if } n \text{ be odd.} \end{cases}$$

Therefore, whether $n$ be even or odd,

$$\cos A = \cos^2 \tfrac{1}{2}A - \sin^2 \tfrac{1}{2}A.$$

II. If the angle A be contained in the second expression, then $\cos A = \cos(\pm n.360^\circ + 180^\circ \pm a) = -\cos a.$

$$\cos \tfrac{1}{2}A = \cos(\pm n.180^\circ + 90^\circ \pm \tfrac{1}{2}a)$$
$$= \begin{cases} \cos(90^\circ \pm \tfrac{1}{2}a) = \mp \sin \tfrac{1}{2}a, \text{ when } n \text{ is even,} \\ -\cos(90^\circ \pm \tfrac{1}{2}a) = \pm \sin \tfrac{1}{2}a, \text{ when } n \text{ is odd.} \end{cases}$$
$$\sin \tfrac{1}{2}A = \sin(\pm n.180^\circ + 90^\circ \pm \tfrac{1}{2}a)$$
$$= \begin{cases} \sin(90^\circ \pm \tfrac{1}{2}a) = \cos \tfrac{1}{2}a, \text{ when } n \text{ is even,} \\ -\sin(90^\circ \pm \tfrac{1}{2}a) = -\cos \tfrac{1}{2}a, \text{ when } n \text{ is odd.} \end{cases}$$

Therefore, whether $n$ be even or odd,

$$-\cos A = \sin^2 \tfrac{1}{2}A - \cos^2 \tfrac{1}{2}A, \text{ or } \cos A = \cos^2 \tfrac{1}{2}A - \sin^2 \tfrac{1}{2}A.$$

## EXTENSION OF THE FORMULÆ FOR ANGULAR FUNCTIONS OF THE SUM OR DIFFERENCE OF ANGLES.

112. *The formulæ*

$$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A,$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B,$$

[proved (Lesson IV., Example I, and Exercise 11) for angles less than 90°, and whose sum does not exceed 90°], *are true for all angles.*

First, if true for positive angles, they will be true for negative angles.

Let $-A$ and $-B$ be negative angles, A and B being positive. Then $(-A)+(-B)$ will be equal to $-(A+B)$, and $(-A)-(-B)$ to $-(A-B)$.

Then, $\sin[(-A)+(-B)] = -\sin(A+B)$
$$= -[\sin A \cos B + \sin B \cos A]$$
$$\sin[(-A)-(-B)] = -\sin(A-B)$$
$$= -[\sin A \cos B - \sin B \cos A]$$
$$\cos[(-A)+(-B)] = \cos(A+B)$$
$$= \cos A \cos B - \sin A \sin B$$
$$\cos[(-A)-(-B)] = \cos(A-B)$$
$$= \cos A \cos B + \sin A \sin B.$$

But, $\sin A = -\sin(-A)$, $\cos A = \cos(-A)$,

$\sin B = -\sin(-B)$, $\cos B = \cos(-B)$,

Therefore, $\sin[(-A)+(-B)]$

$$= -[-\sin(-A)\cos(-B) - \sin(-B)\cos(-A)]$$

$$= \sin(-A)\cos(-B) + \sin(-B)\cos(-A).$$

And similarly for the other formulæ.

Secondly, these formulæ will be true for all positive angles.

I. If the two angles A and B be each less than 90°, but their sum greater than 90°, and not greater than 180°.

Let $A' = 90° - A$, and $B' = 90° - B$; then each of the three angles $A'$, $B'$, and $A'+B'$, is less than 90°, and consequently the formulæ being proved for such angles,

$$\sin(A'+B') = \sin A' \cos B' + \sin B' \cos A',$$

$$\cos(A'+B') = \cos A' \cos B' - \sin A' \sin B'.$$

But $\sin A' = \cos A$, $\cos A' = \sin A$,

$\sin B' = \cos B$, $\cos B' = \sin B$,

and, because $A'+B' = 180° - (A+B)$,

$$\sin(A'+B') = \sin(A+B), \quad \cos(A'+B') = -\cos(A+B).$$

The substitution of these values gives,

$$\sin(A+B) = \sin A \cos B + \sin B \cos A,$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

Again,

$$\sin(A-B) = \sin(B'-A'), \quad \cos(A-B) = \cos(B'-A'),$$

Hence,

$$\sin(A-B) = \sin B' \cos A' - \sin A' \cos B'$$

$$= \cos B \sin A - \cos A \sin B,$$

$$\cos(A-B) = \cos B' \cos A' + \sin B' \sin A'$$

$$= \sin B \sin A + \cos B \cos A.$$

II. If the positive angles A and B be any whatever, and $\alpha$ and $\beta$ the angles not greater than 90°, whose angular functions are equal in absolute value to those of A and B, the angle A will be contained in one of the formulæ,

$$n.360° \pm \alpha, \text{ or } n.360° + 180° \pm \alpha,$$

and B in one of the formulæ,

$$m.360° \pm \beta, \text{ or } m.360° + 180° \pm \beta.$$

The angle A+B will, therefore, be contained in one of the formulæ,

$$p.360° \pm \alpha \pm \beta, \text{ or } p.360° + 180 \pm \alpha \pm \beta.$$

In the first case

(a) $\sin(A+B)=\sin(\pm\alpha\pm\beta)$

$$=\sin(\pm\alpha)\cos(\pm\beta)+\sin(\pm\beta)\cos(\pm\alpha)$$

and in the second,

(b) $\sin(A+B)=-\sin(\pm\alpha\pm\beta)$

$$=-[\sin(\pm\alpha)\cos(\pm\beta)+\sin(\pm\beta)\cos(\pm\alpha)].$$

Now, in the first case,

$$A-n.360^\circ=\pm\alpha$$
$$B-m.360^\circ=\pm\beta$$

whence $\begin{cases}\sin(\pm\alpha)=\sin A, & \cos(\pm\alpha)=\cos A,\\ \sin(\pm\beta)=\sin B, & \cos(\pm\beta)=\cos B;\end{cases}$

or $A-n.360^\circ-180^\circ=\pm\alpha$

$$B-m.360^\circ-180^\circ=\pm\alpha$$

whence $\begin{cases}\sin(\pm\alpha)=-\sin A, & \cos(\pm\alpha)=-\cos A,\\ \sin(\pm\beta)=-\sin B, & \cos(\pm\beta)=-\cos B.\end{cases}$

Therefore, by substitution in (a),

$$\sin(A+B)=\sin A\cos B+\sin B\cos A.$$

In the second case,

$$A-n.360^\circ=\pm\alpha$$
$$B-m.360^\circ-180^\circ=\pm\beta$$

whence $\begin{cases}\sin(\pm\alpha)=\sin A, & \cos(\pm\alpha)=\cos A,\\ \sin(\pm\beta)=-\sin B, & \cos(\pm\beta)=-\cos B;\end{cases}$

or $A-n.360^\circ-180^\circ=\pm\alpha$

$$B-m.360^\circ=\pm\beta$$

whence $\begin{cases}\sin(\pm\alpha)=-\sin A, & \cos(\pm\alpha)=-\cos A,\\ \sin(\pm\beta)=\sin B, & \cos(\pm\beta)=\cos B,\end{cases}$

and, by substitution in (b),

$$\sin(A+B)=\sin A\cos B+\sin B\cos A.$$

It may be shown, in the same way, that the other three formulæ extend to the case where the angles are any whatever.

113. It is to be observed that from any one of these four formulæ, the other three can be very readily deduced.

114. From the formulæ for the sine and cosine of the sum, or the difference of two angles, corresponding formulæ for the tangent of the sum, or of the difference of two angles, can be derived by division.

Thus, $\tan(A\pm B)=\dfrac{\sin(A\pm B)}{\cos(A\pm B)}=\dfrac{\sin A\cos B\pm\sin B\cos A}{\cos A\cos B\mp\sin A\sin B}$

$$=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}.$$

## IMPORTANT FORMULÆ DERIVED FROM THE FOREGOING.

115. Expressions for the sum or the difference of the sines and the cosines of two angles, in terms of angular functions of the half sum and half difference of the angles, may be obtained from the foregoing formulæ.

By addition and subtraction,

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B$$
$$\sin(A+B)-\sin(A-B)=2\sin B\cos A$$
$$\cos(A+B)+\cos(A-B)=2\cos A\cos B$$
$$\cos(A-B)-\cos(A+B)=2\sin A\sin B$$

Let $\qquad A+B=A',\ A-B=B',$

whence $\qquad A=\frac{1}{2}(A'+B'),\ B=\frac{1}{2}(A'-B').$

Then, by substitution,

$$\sin A'+\sin B'=2\sin\tfrac{1}{2}(A'+B')\cos\tfrac{1}{2}(A'-B')$$
$$\sin A'-\sin B'=2\sin\tfrac{1}{2}(A'-B')\cos\tfrac{1}{2}(A'+B')$$
$$\cos A'+\cos B'=2\cos\tfrac{1}{2}(A'+B')\cos\tfrac{1}{2}(A'-B')$$
$$\cos B'-\cos A'=2\sin\tfrac{1}{2}(A'+B')\sin\tfrac{1}{2}(A'-B')$$

or, omitting the dashes,

$$\sin A+\sin B=2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$
$$\sin A-\sin B=2\sin\tfrac{1}{2}(A-B)\cos\tfrac{1}{2}(A+B)$$
$$\cos A+\cos B=2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$
$$\cos B-\cos A=2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$

116. The process by which each of the following formulæ is derived from those just established, is obvious.

$$\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$$
$$\frac{\sin A+\sin B}{\cos A+\cos B}=\tan\tfrac{1}{2}(A+B)$$
$$\frac{\sin A+\sin B}{\cos B-\cos A}=\cot\tfrac{1}{2}(A-B)$$
$$\frac{\sin A-\sin B}{\cos A+\cos B}=\tan\tfrac{1}{2}(A-B)$$
$$\frac{\sin A-\sin B}{\cos B-\cos A}=\cot\tfrac{1}{2}(A+B)$$
$$\frac{\cos A+\cos B}{\cos B-\cos A}=\cot\tfrac{1}{2}(A+B)\cot\tfrac{1}{2}(A-B)$$
$$\tan A\pm\tan B=\frac{\sin(A\pm B)}{\cos A\cos B}$$
$$\sin(A+B)\sin(A-B)=\begin{cases}\sin^2A-\sin^2B\\ \cos^2B-\cos^2A\end{cases}$$

# LESSON XIII.

*Elementary formulæ of Analytical Trigonometry, continued.*—Formulæ for angular functions of multiple and sub-multiple angles—The formulæ used in the solution of triangles derived the one from the other.

## FORMULÆ FOR ANGULAR FUNCTIONS OF MULTIPLE AND SUB-MULTIPLE ANGLES.

117. The formulæ

$$\begin{cases} \sin(m+1)A = 2\sin mA \cos A - \sin(m-1)A \\ \cos(m+1)A = 2\cos mA \cos A - \cos(m-1)A, \end{cases}$$

known as Thomas Simpson's formulæ, are derived from those for the sine and the cosine of the sum and the difference of two angles, by changing A into $mA$, and B into A, in the formulæ (112), and then adding the first two results together, and also the other two results.

The formulæ

$$\begin{cases} \sin(m+1)A = 2\cos mA \sin A + \sin(m-1)A \\ \cos(m+1)A = \cos(m-1)A - 2\sin mA \sin A \end{cases}$$

may be obtained by a similar process.

118. By making $m=1$, these formulæ give

$$\sin 2A = 2\sin A\cos A,\ \cos 2A = \begin{cases} 2\cos^2 A - 1, \\ 1 - 2\sin^2 A. \end{cases}$$

By making $m=2$,

$$\sin 3A = 3\sin A - 4\sin^3 A, \text{ and } \cos 3A = 4\cos^3 A - 3\cos A.$$

For,

$$\begin{aligned} \sin 3A &= 2\sin 2A\cos A - \sin A \\ &= 4\sin A\cos^2 A - \sin A \\ &= 4\sin A(1-\sin^2 A) - \sin A \\ &= 3\sin A - 4\sin^3 A. \\ \cos 3A &= 2\cos 2A\cos A - \cos A \\ &= 2\cos A(2\cos^2 A - 1) - \cos A \\ &= 4\cos^3 A - 3\cos A. \end{aligned}$$

In this way by successive substitutions values for $\sin 4A$, $\sin 5A, \ldots\ldots$ $\cos 4A$, $\cos 5A, \ldots\ldots$ may be obtained.

119. *Given the sine of an angle, to find the sine and the cosine of the double of the angle, and also of its triple.*

Let A be the angle whose sine is given, it is required to find $\sin 2A$ and $\cos 2A$, $\sin 3A$ and $\cos 3A$.

$$\sin 2A = \pm 2\sin A\sqrt{(1-\sin^2 A)},\ \cos 2A = 1 - 2\sin^2 A$$

$$\begin{cases} \sin 3A = 3\sin A - 4\sin^3 A, \\ \cos 3A = (1 - 4\sin^2 A)[\pm\sqrt{(1-\sin^2 A)}]. \end{cases}$$

From these values it will be seen that cos 2A and sin 3A are fully determined, while the absolute values alone can be found for sin 2A and cos 3A.

120. *Given the cosine of an angle, to find the sine and the cosine of the double of the angle, and of its triple.*

$$\sin 2A = \pm\, 2\cos A\sqrt{(1-\cos^2 A)},\ \cos 2A = 2\cos^2 A - 1$$

$$\begin{cases} \sin 3A = [\pm\sqrt{(1-\cos^2 A)}](4\cos^2 A - 1), \\ \cos 3A = 4\cos^3 A - 3\cos A \end{cases}$$

The absolute values alone of sin 2A and sin 3A can be found from these expressions; but cos 2A and cos 3A are completely determined.

121. *Given the sine of an angle, to find the sine and the cosine of its half.*

Let A represent the angle whose sine $\frac{p}{q}$ is given, it is required to find the values of $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$.

The formulæ

(39) $$\sin^2\tfrac{1}{2}A + \cos^2\tfrac{1}{2}A = 1$$

(41) $$2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A = \sin A,$$

give, by addition and subtraction,

$$(\sin\tfrac{1}{2}A + \cos\tfrac{1}{2}A)^2 = 1 + \sin A,$$

$$(\sin\tfrac{1}{2}A - \cos\tfrac{1}{2}A)^2 = 1 - \sin A,$$

whence,

$$\sin\tfrac{1}{2}A + \cos\tfrac{1}{2}A = \pm\sqrt{(1+\sin A)},$$

$$\sin\tfrac{1}{2}A - \cos\tfrac{1}{2}A = \pm\sqrt{(1-\sin A)},$$

and

$$2\sin\tfrac{1}{2}A = \pm\sqrt{(1+\sin A)} \pm \sqrt{(1-\sin A)},$$

$$= \pm\sqrt{\left(1+\frac{p}{q}\right)} \pm \sqrt{\left(1-\frac{p}{q}\right)}$$

$$2\cos\tfrac{1}{2}A = \pm\sqrt{(1+\sin A)} \mp \sqrt{(1-\sin A)},$$

$$= \pm\sqrt{\left(1+\frac{p}{q}\right)} \mp \sqrt{\left(1-\frac{p}{q}\right)}.$$

When the sine of an angle is given, there are, therefore, four values for the sine, and four for the cosine of its half; but among the values for the sine of half the angle are to be found the values for the cosine of the half, and *vice versâ*.

To understand why there should be four values for the sine and for the cosine of half an angle in function of the sine of the angle, and why the four values for the sine

should be the same as those for the cosine, it must be borne in mind that all positive angles having $\frac{p}{q}$ for their sine are contained in the expressions

(109) $$n.360^\circ+a,\quad n.360^\circ+180^\circ-a,$$

in which $a$ represents the angle less than 90° that has $\frac{p}{q}$ for its sine.

Hence the required angular functions will be expressed by

$$\sin\tfrac{1}{2}A=\sin(\tfrac{1}{2}n.360^\circ+\tfrac{1}{2}a),$$

or $$\sin\tfrac{1}{2}A=\sin(\tfrac{1}{2}n.360^\circ+90^\circ-\tfrac{1}{2}a)$$

$$\cos\tfrac{1}{2}A=\cos(\tfrac{1}{2}n.360^\circ+\tfrac{1}{2}a),$$

or $$\cos\tfrac{1}{2}A=\cos(\tfrac{1}{2}n.360^\circ+90^\circ-\tfrac{1}{2}a).$$

Now, for all even values of $n$, these become respectively,

$$\sin\tfrac{1}{2}A=\sin\tfrac{1}{2}a,\text{ or }\sin\tfrac{1}{2}A=\cos\tfrac{1}{2}a$$

$$\cos\tfrac{1}{2}A=\cos\tfrac{1}{2}a,\text{ or }\cos\tfrac{1}{2}A=\sin\tfrac{1}{2}a,$$

and for the odd values of $n$,

$$\sin\tfrac{1}{2}A=\sin(180^\circ+\tfrac{1}{2}a)=-\sin\tfrac{1}{2}a,$$

or $$\sin\tfrac{1}{2}A=-\sin(90^\circ-\tfrac{1}{2}a)=-\cos\tfrac{1}{2}a$$

$$\cos\tfrac{1}{2}A=\cos(180^\circ+\tfrac{1}{2}a)=-\cos\tfrac{1}{2}a$$

or $$\cos\tfrac{1}{2}A=-\cos(90^\circ-\tfrac{1}{2}a)=-\sin\tfrac{1}{2}a.$$

The mode of investigation will be the same in the case of negative angles.

122. *Given the cosine of an angle, to find the sine and the cosine of its half.*

Let A be the angle whose cosine $\frac{p}{q}$ is given, it is required to find values for $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$.

From the formulæ (118) are deduced the values

$$\pm\sqrt{\left(\frac{q-p}{2q}\right)}\text{ for }\sin\tfrac{1}{2}A,\text{ and }\pm\sqrt{\left(\frac{q+p}{2q}\right)}\text{ for }\cos\tfrac{1}{2}A.$$

The cause of the ambiguity of sign exhibited in these values, will be seen from considerations analogous to those in the foregoing problem.

123. *Given the sine of an angle, to find the sine of its third part.*

Let A be the angle whose sine $\frac{p}{q}$ is given.

The formula (118), $\sin 3A = 3 \sin A - 4 \sin^3 A$, may be written

$$\sin^3 \tfrac{1}{3}A - \tfrac{3}{4} \sin \tfrac{1}{3}A + \tfrac{1}{4} \sin A = 0,$$

or

$$\sin^3 \tfrac{1}{3}A - \tfrac{3}{4} \sin \tfrac{1}{3}A + \tfrac{1}{4}\frac{p}{q} = 0;$$

and the three roots of this cubic equation will be the required values of $\sin \frac{1}{3}A$.

If in the cubic equation, $x^3 + qx + r = 0$, the coefficient $q$ be negative, and $(\frac{1}{3}q)^3$ be greater than $(\frac{1}{2}r)^2$, the three roots are real, and the equation belongs to what has been called, because Cardan's method of solution fails, the irreducible case. Now in the equation from which the values of $\sin \frac{1}{3}A$ are to be obtained, the conditions pertaining to the irreducible case exist; there are, therefore, three values for the sine of the third of an angle whose sine is given.

That there should be three values, and only three, for $\sin \frac{1}{3}A$, when the sine of A is given, will appear from the following considerations:

When an angle is given by its sine, it is indeterminate, and, if positive, may be any of the angles in one of the expressions

$$n.360° + a \qquad\qquad n.360° + 180° - a;$$

consequently the sine of its third part will be expressed by

$$\sin(\tfrac{1}{3}n.360° + \tfrac{1}{3}a) \text{ or } \sin(\tfrac{1}{3}n.360° + \tfrac{1}{3}\,180° - \tfrac{1}{3}a).$$

Now $n$ will be a multiple of 3, or exceed a multiple of 3 by 1 or by 2, that is, $n$ will be of the form $3m$, or $3m+1$, or $3m+2$. Hence the sine of $\frac{1}{3}A$ will be given by one of the six formulæ

$$\sin(m.360° + \tfrac{1}{3}a) \qquad \sin(m.360° + \tfrac{1}{3}.180° - \tfrac{1}{3}a)$$
$$\sin(m.360° + \tfrac{1}{3}.360° + \tfrac{1}{3}a), \sin(m.360° + \tfrac{1}{3}.360° + \tfrac{1}{3}.180° - \tfrac{1}{3}a)$$
$$\sin(m.360° + \tfrac{2}{3}.360° + \tfrac{1}{3}a), \sin(m.360° + \tfrac{2}{3}.360° + \tfrac{1}{3}.180° - \tfrac{1}{3}a)$$

or,

(1) $\sin \frac{1}{3}a$ (4) $\sin(\frac{1}{3}.180° - \frac{1}{3}a)$

(2) $\sin(\frac{1}{3}.360° + \frac{1}{3}a)$ (5) $\sin(180° - \frac{1}{3}a)$

(3) $\begin{cases} \sin(\frac{2}{3}.360° + \frac{1}{3}a) \\ \text{or} - \sin(\frac{1}{3}.180° + \frac{1}{3}a) \end{cases}$ (6) $\begin{cases} \sin(\frac{2}{3}.360° + \frac{1}{3}.180° - \frac{1}{3}a) \\ \text{or} - \sin(\frac{2}{3}.180° - \frac{1}{3}a). \end{cases}$

The angles in (1) and (5), (2) and (4), (3) and (6), are supplements, and, therefore, their sines are equal. There are therefore three values, and only three, for $\sin \frac{1}{3}A$ in function of $\sin A$.

124. *Given the cosine of an angle, to find the cosine of its third part.*

Let A be the angle whose cosine $\frac{p}{q}$ is given.

The roots of the cubic equation

$$\cos^3 \tfrac{1}{3}A - \tfrac{3}{4}\cos \tfrac{1}{3}A - \tfrac{1}{4}\cos A = 0,$$

or,

$$\cos^3 \tfrac{1}{3}A - \tfrac{3}{4}\cos \tfrac{1}{3}A - \tfrac{1}{4}\frac{p}{q} = 0,$$

derived from formula (118), will be the values of the required cosine.

This cubic equation belongs to the irreducible case; consequently its three roots are real. There are, therefore, three values for the cosine of the third of an angle in function of the cosine of the angle.

This can be accounted for by a process similar to that employed in the preceding problem.

### THE FORMULÆ USED IN THE SOLUTION OF TRIANGLES DERIVED THE ONE FROM THE OTHER.

125. The chief formulæ for the solution of plane triangles may be classed in three groups:

$$[1]\begin{cases} a = b\cos C + c\cos B \\ b = c\cos A + a\cos C \\ c = a\cos B + b\cos A. \end{cases} \quad [2]\begin{cases} a^2 = b^2 + c^2 - 2bc\cos A \\ b^2 = c^2 + a^2 - 2ca\cos B \\ c^2 = a^2 + b^2 - 2ab\cos C. \end{cases}$$

$$[3]\ \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

These groups, however, are not independent the one of the other; but any two of them are included in the third.

I. For instance, the elimination of cos B and cos C among the three equations of group [1], leads to the first equation in group [2]. The remaining equations of this group will in a similar way result from the elimination of cos C and cos A, and then of cos A and cos B among the three equations [1].

Again, the elimination of cos C and $c$ among the equations of group [1] gives

$$\begin{aligned} a^2 - b^2 &= c(a\cos B - b\cos A) \\ &= (a\cos B + b\cos A)(a\cos B - b\cos A \\ &= a^2\cos^2 B - b^2\cos^2 A. \end{aligned}$$

Hence,

$$a^2(1 - \cos^2 B) = b^2(1 - \cos^2 A), \text{ and } \frac{\sin^2 A}{a^2} = \frac{\sin^2 B}{b^2}.$$

Extracting the square root, and observing that in consequence of the angles A and B being less than 180°, their sines must be positive, there results $\frac{\sin A}{a}=\frac{\sin B}{b}$.

II. That the expressions in group [1] can be derived from group [2] will appear by adding together two of the equations in group [2], cancelling the common terms, transposing, and dividing by the common factor.

That the equations [3] can also be derived from those in group [2] may be shown as follows.

The formula $a^2=b^2+c^2-2bc\cos A$

gives $4b^2c^2\cos^2 A=(b^2+c^2-a^2)^2$

whence $4b^2c^2(1-\cos^2 A)=4b^2c^2-(b^2+c^2-a^2)^2$

and $$\frac{\sin^2 A}{a^2}=\frac{2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4}{4a^2b^2c^2}.$$

The same value would be found for $\frac{\sin^2 B}{b^2}$, and for $\frac{\sin^2 C}{c^2}$; for the second side of the last equation does not change in value, when $b$ is substituted for $a$ and $a$ for $b$, or when $c$ is substituted for $a$ or for $b$, and $a$ or $b$ for $c$.

Therefore $$\frac{\sin^2 A}{a^2}=\frac{\sin^2 B}{b^2}=\frac{\sin^2 C}{c^2},$$

and since the angles A, B, C, are less than 180°,

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}.$$

III. It remains to be shown how the formulæ in [1] and [2] may be deduced from the equations [3].

The equation $$\frac{\sin B}{b}=\frac{\sin C}{c}$$

gives $$\frac{\sin B\cos C}{b\cos C}=\frac{\sin C\cos B}{c\cos B},$$

whence $$\frac{\sin B\cos C+\sin C\cos B}{b\cos C+c\cos B}=\frac{\sin C}{c}=\frac{\sin A}{a},$$

that is, $$\frac{\sin(B+C)}{b\cos C+c\cos B}=\frac{\sin A}{a},$$

but $\sin A=\sin(B+C)$, because $A+B+C=180°$,

therefore $b\cos C+c\cos B=a$.

In this way also the other two equations [1] may be obtained.

Lastly, the relations [2] are contained in the equations [3].

For from $$\frac{\sin^2 A}{a^2}=\frac{\sin^2 B}{b^2}=\frac{\sin^2 C}{c^2},$$

it follows that $$\frac{\sin^2 B+\sin^2 C-\sin^2 A}{b^2+c^2-a^2}=\frac{\sin^2 A}{a^2},$$

whence $$b^2+a^2-a^2=\frac{a^2(\sin^2 B+\sin^2 C-\sin^2 A)}{\sin^2 A}.$$

But when three angles A, B, and C, are the three angles of a plane triangle, $\cos A=\dfrac{\sin^2 B+\sin^2 C-\sin^2 A}{2\sin B\sin C}$. Therefore

$$b^2+c^2-a^2=2\cos A.\frac{a\sin B}{\sin A}.\frac{a\sin C}{\sin A}=2bc\cos A,$$

and $$a^2=b^2+c^2-2bc\cos A.$$

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# LESSON XIV.

*Elementary formulæ of Analytical Trigonometry, continued:*—Circular functions—Circular measure of an angle, of an arc—Solution of Quadratic and Cubic Equations by means of trigonometrical functions.

### CIRCULAR FUNCTIONS.

126. If in a circle the arc subtending at the centre the unit of angular measure be taken as the standard for the arcs of this circle, the measure of any angle at the centre, and that of the arc of the circle on which it stands, will be expressed by the same number. For in the same circle the angles at the centre are to one another as the arcs on which they stand (89, 4).

Moreover, as the arcs which in different circles subtend the same, or equal, angles at the centre, are to one another as the circumferences of their respective circles, and as consequently the measures of these arcs are as the measures of the circumferences, the value of the ratios of several arcs of different radii to their circumferences will not alter so long as the angle subtended by them remains the same.

Hence to an angle at the centre which is the $\frac{2}{3}$ of a right angle, will always correspond an arc the $\frac{2}{3}$ of the fourth part of the circumference, whatever may be the size of the circumference of which it is a part. In other words, the arc and the angle which it subtends at the centre will both be expressed by 60°.

Arcs thus expressed in degrees, minutes, seconds, that is, arcs measured by the 360th part of their circumferences, may therefore be substituted in investigations for the angles that stand on them, and the definitions and properties of angular functions may be transferred to the analogous functions of corresponding arcs. These functions are then called *circular functions.*

127. When considered as a circular function, the sine will be defined to be the ratio which a straight line drawn from one extremity of an arc at right angles to the diameter passing through the other extremity, bears to the radius of the circle.

This ratio is equal to that which would be the sine of the angle at the centre standing upon the arc.

And so on for the other functions.

128. It is also evident that all the relations established among the angular functions will be the same for the circular functions of arcs expressed in degrees, minutes, &c.

129. Sometimes the sine of an arc is said to be the straight line drawn from one extremity of the arc at right angles to the diameter passing through the other extremity; the tangent of an arc is said to be the part of the tangent at one extremity intercepted between the diameters passing through the two extremities of the arc; the cosine of an arc is said to be the part of the diameter between the centre of the circle and the sine of the arc. These definitions are not, however, accurate, unless by the finite portion of the straight line mentioned in each is to be understood its measure when referred to the radius of the arc as unit of lineal measure. With this restriction these functions become really *ratios* or numbers, and wholly agree with the angular and circular functions as already defined; without it, arcs of the same number of degrees, belonging to circles of different radii, would have different sines, different tangents, different cosines.

### CIRCULAR MEASURE OF AN ANGLE, OF AN ARC.

130. A unit of angular measure different from that adopted in the foregoing investigations, is frequently resorted to, especially in the higher branches of Mathematics.

Instead of comparing any given angle with the degree, or the 90th part of a right angle, it is compared with the angle which is the $\pi$th part of two right angles, and is expressed by $57^\circ{\cdot}29577$......to the fifth decimal place, or by $206265''$ nearly.

Let $u^\circ$ represent the value in degrees of this new angular unit, and let $A^\circ$ be the value in degrees of any given angle A; then the number obtained by dividing $A^\circ$ by $u^\circ$ will be the measure of the angle A referred to the new unit of measure. This number which is the quotient obtained by dividing the degrees and parts of a degree in a given angle by $u^\circ$, or $57^\circ\cdot29577\ldots$, is called the *circular* measure of the angle.

Thus the angle of $27^\circ\ 14'\ 37''$ will have the fraction $\frac{3724323}{5729577}$, or $\cdot475$ for its circular measure.

131. To understand the reason of this name, it will be sufficient to consider that, if in a circle whose radius is $r$, $a$ be the *length* of an arc containing $A^\circ$, that is, subtending at the centre an angle of $A^\circ$, the ratio of $A^\circ$ to $180^\circ$ shall be the same as that of $a$ to $\pi r$, whence

$$A^\circ = \frac{180^\circ}{\pi}.\frac{a}{r}, \text{ or } A^\circ : \frac{180^\circ}{\pi} = a : r, \text{ or } \frac{A^\circ}{u^\circ} = \frac{a}{r},$$

that is to say, the number expressing the quotient of $A^\circ$ by $57^\circ\cdot29577\ldots$ shall be the same as the number expressing the length of the arc referred to its radius as unit of lineal measure.

Thus, if the length of an arc be 15 inches, and its radius be 3 feet, $\frac{5}{12}$ will be the circular measure of this arc, and $\frac{5}{12}\times 57\cdot29577$ will be the number of degrees it contains.

132. It further appears that if an arc contain $u^\circ$, its length will be equal to the radius of the circle to which it belongs.

When the arc equal in length to the radius is taken for the unit of measure of the arcs in the same circle, the measure of an arc, referred to this standard, is said to be its *circular* measure.

133. The formulæ hitherto established, connecting angular functions the one with the other, will all hold if the angles be expressed by their circular measures instead of the degrees, minutes, and seconds they contain. So also will the corresponding formulæ for the circular functions of arcs.

The circular measures of angles and arcs are usually denoted by the letters of the Greek alphabet.

134. The circular measure of an angle or an arc of $180^\circ$ is $\pi$; for $\frac{180^\circ}{u^\circ} = \frac{\pi r}{r} = \pi$.

## QUADRATIC EQUATIONS.

135. *To solve an equation of the second degree by means of angular functions.*

Every algebraic equation of the second degree can be brought to the form $x^2+px+q=0$, $p$ and $q$ being integral or fractional, positive or negative; and the roots $x'$ and $x''$ of this equation will have for their expression

$$x'=-\tfrac{1}{2}p\left[1-\left(1-\frac{4q}{p^2}\right)^{\frac{1}{2}}\right],\qquad x''=-\tfrac{1}{2}p\left[1+\left(1-\frac{4q}{p^2}\right)^{\frac{1}{2}}\right].$$

As the absolute values of $x'$ and $x''$ are independent of the sign of the coefficient $p$, it will be sufficient to discuss the case in which $q$ is positive and that in which it is negative.

136. I. Let $q$ be positive. When $4q$ is less than $p^2$, both roots will be real.

Find an acute angle $\phi$ such that $\sin\phi=\dfrac{2q^{\frac{1}{2}}}{p}$. The values of $x'$ and $x''$ become then

$$x'=-p\sin^2\tfrac{1}{2}\phi,\qquad x''=-p\cos^2\tfrac{1}{2}\phi,$$

or

$$x'=-q^{\frac{1}{2}}\tan\tfrac{1}{2}\phi,\qquad x''=-q^{\frac{1}{2}}\cot\tfrac{1}{2}\phi.$$

These values are calculable by logarithms.

When $4q$ is greater than $p^2$, the roots are imaginary, but they may be expressed by means of angular functions, and the portions which are real computed. For the roots may be written

$$x'=-\tfrac{1}{2}p\left[1-(-1)^{\frac{1}{2}}\left(\frac{4q}{p^2}-1\right)^{\frac{1}{2}}\right]$$

$$x''=-\tfrac{1}{2}p\left[1+(-1)^{\frac{1}{2}}\left(\frac{4q}{p^2}-1\right)^{\frac{1}{2}}\right]$$

Let $\theta$ be an acute angle such that $\sec\theta=\dfrac{2q^{\frac{1}{2}}}{p}$, then

$$x'=-\tfrac{1}{2}p(1-(-1)^{\frac{1}{2}}\tan\theta)\qquad x''=-\tfrac{1}{2}p(1+(-1)^{\frac{1}{2}}\tan\theta).$$

137. II. Let $q$ be negative. The roots will then be always real, whatever may be the relative magnitudes of $p$ and $q$. They may be written,

$$x'=-\tfrac{1}{2}p\left[1-\left(1+\frac{4q}{p^2}\right)^{\frac{1}{2}}\right],\quad x''=-\tfrac{1}{2}p\left[1+\left(1+\frac{4q}{p^2}\right)^{\frac{1}{2}}\right]$$

Find $a$ such that $\tan a = \frac{2q^{\frac{1}{2}}}{p}$; then,

$$x' = -\tfrac{1}{2}p[1 - \sec a] \qquad x'' = -\tfrac{1}{2}p[1 + \sec a]$$
$$= p\frac{\sin^2 \frac{1}{2}a}{\cos a} \qquad = -p\frac{\cos^2 \frac{1}{2}a}{\cos a},$$

or $\quad x' = q^{\frac{1}{2}} \tan \tfrac{1}{2}a \qquad x'' = -q^{\frac{1}{2}} \cot \tfrac{1}{2}a$

formulæ to which logarithmic computation can be immediately applied.

## CUBIC EQUATIONS.

138. *To solve an equation of the third degree by means of angular functions.*

Every equation of the third degree can be reduced to the form $x^3 + qx + r = 0$, in which $q$ and $r$ may be positive or negative, integral or fractional. According to Cardan's method of solving cubic equations, the value of $x$ in this equation will be found to be made up of two parts, $z'$ and $z''$, whose cubes $z'^3$ and $z''^3$ are the roots of the equation

$$z^2 + rz - \frac{1}{27}q^3 = 0,$$

and have for their values,

$$z'^3 = -\tfrac{1}{2}r\left[1 - \left(1 + \frac{4q^3}{27r^2}\right)^{\frac{1}{2}}\right], \quad z''^3 = -\tfrac{1}{2}r\left[1 + \left(1 + \frac{4q^3}{27r^2}\right)^{\frac{1}{2}}\right].$$

But the cube roots of $z'^3$ and $z''^3$ are,

$$(1)\ z', \quad (2) -\tfrac{1}{2}z'[1 + (-3)^{\frac{1}{2}}], \quad (3) -\tfrac{1}{2}z'[1 - (-3)^{\frac{1}{2}}]$$
$$(4)\ z'', \quad (5) -\tfrac{1}{2}z''[1 + (-3)^{\frac{1}{2}}], \quad (6) -\tfrac{1}{2}z''[1 - (-3)^{\frac{1}{2}}],$$

giving 3 values for each of the parts $z'$ and $z''$, which have to be added together for determining each of the roots of the cubic equation. In combining a value of $z'$ with a value of $z''$, it is to be remembered that those only can be combined whose product is real; for in the course of the operations it has been supposed that $z'z''$ was equal to $-\frac{1}{3}q$, and therefore a real quantity. It is, therefore, only the values (1) and (4), (2) and (6), (3) and (5), which can be taken together for the roots of the given cubic equation.

These roots will, therefore, be

$$x' = z' + z''$$
$$\left.\begin{array}{l} x'' \\ x''' \end{array}\right\} = -\tfrac{1}{2}(z' + z'') \mp \tfrac{1}{2}(z' - z'')(-3)^{\frac{1}{2}}.$$

The absolute values of the roots being independent of the sign of $r$, it will be sufficient to examine the case where, in the given cubic equation, the coefficient $q$ is positive, and that in which it is negative.

139. I. Let $q$ be positive. Then $x'$ shall be real, $x''$ and $x'''$ imaginary.

Find the acute angle $\phi$, such that $\tan\phi=\frac{2}{r}\left(\frac{q^3}{27}\right)^{\frac{1}{2}}$.

Then

$$\begin{aligned} z' &= \{-\tfrac{1}{2}r[1-(1+\tan^2\phi)]^{\frac{1}{2}}\}^{\frac{1}{3}} \\ &= [-\tfrac{1}{2}r(1-\sec\phi)]^{\frac{1}{3}} \\ &= \left[\frac{r}{2\cos\phi}(1-\cos\phi)\right]^{\frac{1}{3}} \\ &= \left[\left(\frac{q^3}{27}\right)^{\frac{1}{2}}\frac{1-\cos\phi}{\sin\phi}\right]^{\frac{1}{3}} \\ &= \left[\left(\frac{q^3}{27}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}\phi\right]^{\frac{1}{3}} \\ &= \left(\frac{q}{3}\right)^{\frac{1}{2}}\tan^{\frac{1}{3}}\tfrac{1}{2}\phi \\ &= \left(\frac{q}{3}\right)^{\frac{1}{2}}\tan\psi \end{aligned} \qquad \begin{aligned} z'' &= \{-\tfrac{1}{2}r[1+(1+\tan^2\phi)^{\frac{1}{2}}]\}^{\frac{1}{3}} \\ &= [-\tfrac{1}{2}r(1+\sec\phi)]^{\frac{1}{3}} \\ &= \left[-\frac{r}{2\cos\phi}(1+\cos\phi)\right]^{\frac{1}{3}} \\ &= -\left[\left(\frac{q^3}{27}\right)^{\frac{1}{2}}\frac{1+\cos\phi}{\sin\phi}\right]^{\frac{1}{3}} \\ &= -\left[\left(\frac{q^3}{27}\right)^{\frac{1}{2}}\cot\tfrac{1}{2}\phi\right]^{\frac{1}{3}} \\ &= -\left(\frac{q}{3}\right)^{\frac{1}{2}}\cot^{\frac{1}{3}}\tfrac{1}{2}\phi \\ &= -\left(\frac{q}{3}\right)^{\frac{1}{2}}\cot\psi \end{aligned}$$

by putting $\tan^{\frac{1}{3}}\frac{1}{2}\phi=\tan\psi$.

Hence,
$$x'=\left(\frac{q}{3}\right)^{\frac{1}{2}}(\tan\psi-\cot\psi),$$

$$\left.\begin{matrix}x''\\x'''\end{matrix}\right\}=-\tfrac{1}{2}\left(\frac{q}{3}\right)^{\frac{1}{2}}(\tan\psi-\cot\psi)\mp\tfrac{1}{2}q^{\frac{1}{2}}(-1)^{\frac{1}{2}}(\tan\psi+\cot\psi)$$

or
$$x'=-2\left(\frac{q}{3}\right)^{\frac{1}{2}}\cot 2\psi$$

$$\left.\begin{matrix}x''\\x'''\end{matrix}\right\}=\left(\frac{q}{3}\right)^{\frac{1}{2}}\cot 2\psi\mp q(-1)^{\frac{1}{2}}\operatorname{cosec} 2\psi$$

This value of $x'$ and the real and imaginary parts of the values of $x''$ and $x'''$ can be computed by means of logarithms.

140. II. Let $q$ be negative. The roots of the equation

$$z^2+rz+\frac{1}{27}q^3=0$$

which are

$$\left.\begin{matrix}z'^3\\z''^3\end{matrix}\right\}=-\tfrac{1}{2}r\left[1\mp\left(1-\frac{4q^3}{27r^2}\right)^{\frac{1}{2}}\right]$$

will be real if $27r^2>4q^3$, and imaginary if $27r^2<4q^3$.

If $27r^2 > 4q^3$, let $\theta$ be the acute angle such that $\sin\theta = \frac{2}{r}\left(\frac{q^3}{27}\right)^{\frac{1}{2}}$; then

$$z' = \left\{-\tfrac{1}{2}r(1-\cos\theta)\right\}^{\frac{1}{3}} \qquad z'' = \left\{-\tfrac{1}{2}r(1+\cos\theta)\right\}^{\frac{1}{3}}$$

$$= \left\{-\left(\frac{q^3}{27}\right)^{\frac{1}{2}}\frac{1-\cos\theta}{\sin\theta}\right\}^{\frac{1}{3}} \qquad = \left\{-\left(\frac{q^3}{27}\right)^{\frac{1}{2}}\frac{1+\cos\theta}{\sin\theta}\right\}^{\frac{1}{3}}$$

$$= -\left(\frac{q}{3}\right)^{\frac{1}{2}}\tan^{\frac{1}{3}}\tfrac{1}{2}\theta \qquad = -\left(\frac{q}{3}\right)^{\frac{1}{2}}\cot^{\frac{1}{3}}\tfrac{1}{2}\theta$$

$$= -\left(\frac{q}{3}\right)^{\frac{1}{2}}\tan\psi \qquad = -\left(\frac{q}{3}\right)^{\frac{1}{2}}\cot\psi$$

by making $\tan^{\frac{1}{3}}\frac{1}{2}\theta = \tan\psi$.

Therefore,
$$x' = -\left(\frac{q}{3}\right)^{\frac{1}{2}}(\tan\psi + \cot\psi)$$

$$\left.\begin{matrix} x'' \\ x''' \end{matrix}\right\} = \tfrac{1}{2}\left(\frac{q}{3}\right)^{\frac{1}{2}}(\tan\psi + \cot\psi) \mp \tfrac{1}{2}q^{\frac{1}{2}}(-1)^{\frac{1}{2}}(\cot\psi + \tan\psi)$$

or
$$x' = 2\left(\frac{q}{3}\right)^{\frac{1}{2}}\operatorname{cosec} 2\psi$$

$$\left.\begin{matrix} x'' \\ x''' \end{matrix}\right\} = \left(\frac{q}{3}\right)^{\frac{1}{2}}\operatorname{cosec} 2\psi \mp q^{\frac{1}{2}}(-1)^{\frac{1}{2}}\cot 2\psi$$

By means of these formulæ the real root, and the real parts of the other two roots, may be calculated.

141. If $27r^2 < 4q^3$, the given cubic equation $x^3 - qx + r = 0$ will belong to what is called the irreducible case (123).

Then by making $x = ty$, this equation becomes

$$y^3 - \frac{q}{t^2}y + \frac{r}{t^3} = 0.$$

and comparing it with the value of sin A in function of $\sin\frac{1}{3}$A (118), which may be written

$$\sin^3\tfrac{1}{3}A - \tfrac{3}{4}\sin\tfrac{1}{3}A + \tfrac{1}{4}\sin A = 0,$$

it will appear that if $\frac{q}{t^2}$ were equal to $\frac{3}{4}$, and $\frac{r}{t^3}$ equal to $\frac{1}{4}$ sin A, the roots of the cubic equation in $y$ would be expressed by the values of $\sin\frac{1}{3}$A; and multiplying these by $t$, the results would be the values of $x$ in the given cubic.

Now, if $t$ be taken equal to $2\left(\frac{q}{3}\right)^{\frac{1}{2}}$, and A be an angle such that $\sin A = \frac{r}{2}\left(\frac{27}{q^3}\right)^{\frac{1}{2}}$, which is possible because $27r^2$ is by supposition less than $4q^3$, the coefficients of the equations in $y$ and in $\sin \frac{1}{3}A$ become alike, and, therefore, the three values (123), where $a$ represents the acute angle whose sine is equal to that of A,

$$\sin \tfrac{1}{3}a,\ \sin (60^\circ - \tfrac{1}{3}a),\ -\sin (60^\circ + \tfrac{1}{3}a)$$

will be the roots of the equation in $y$, and

$$2\left(\frac{q}{3}\right)^{\frac{1}{2}} \sin \tfrac{1}{3}a,\ 2\left(\frac{q}{3}\right)^{\frac{1}{2}} \sin (60^\circ - \tfrac{1}{3}a),\quad -2\left(\frac{q}{3}\right)^{\frac{1}{2}} \sin (60^\circ + \tfrac{1}{3}a)$$

will be the roots of the given cubic equation.

### Exercises.

1. $x^3 - \frac{101}{102}x + \frac{19003}{119072} = 0.$
2. $x^3 + 1728\frac{1}{3}x = 123578.$
3. $x^3 + \frac{7}{44}x - \frac{1695}{12716} = 0.$
4. $x^3 - 300x + 1000 = 0.$
5. $x^3 - x^2 - 2x + 1 = 0.$
6. $x^3 + 3x^2 + 3x - 100 = 0.$
7. $x^3 + 10x^2 + 5x = 2600.$

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# LESSON XV.

*Elementary principles of Solid or Spherical Trigonometry.**—Preliminary definitions and propositions—Angular functions of dihedral angles.

## PRELIMINARY DEFINITIONS AND PROPOSITIONS.

142. The angle formed by two planes that meet is a *dihedral* angle.

The two planes that contain it are its *sides* or *faces;* and their common intersection is its *edge.*

A dihedral angle is measured by the angle formed by two straight lines drawn from a point in the edge at right angles to it, one in one side, and the other in the second side of the dihedral angle.

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* Before entering on this branch of Trigonometry, the student should know the first twenty-one propositions of the Eleventh Book of Euclid's Elements, together with Propositions A and B. He may omit Prop. xvii, if he has not yet learned the Fifth and Sixth Books.

The plane of these two straight lines is at right angles to the edge of the dihedral angle.

143. A trihedral angle is a solid angle formed by three planes meeting two and two, and passing through the same point.

This point is the *vertex* of the trihedral angle; the intersections of each pair of planes are its edges; the angles formed in each of the three planes by the edges are the *sides* or *faces;* and the angles formed by these sides are the *dihedral angles* of the trihedral.

If S be the vertex of a trihedral angle, SA, SB, SC, its three edges; the measures in degrees, minutes, seconds, of the dihedral angles shall be represented by the letters A, B, C, and the measures of the sides opposite to them by the corresponding small letters *a*, *b*, *c*. The angles and sides of a trihedral are considered as less than two right angles.

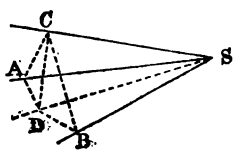
144. A trihedral angle, one of whose dihedral angles is a right angle, is said to be a *right-angled* or *rectangular* trihedral. The side opposite to the right angle is the *hypotenuse.*

If one of the sides or faces be a right angle, the trihedral angle is said to be a *quadrantal* trihedral.

145. *If two of the dihedral angles of a trihedral be right angles, the sides opposite to them shall also be right angles;* and *if the three dihedral angles be right angles, the sides opposite to them shall be right angles.*

In the first case the trihedral is said to be *bi-rectangular,* and in the second *tri-rectangular.*

146. *If two sides of a trihedral angle be equal, the dihedral angles opposite to them shall also be equal;* and *conversely.*




Let the two sides ASC and BSC be equal.

From any point C in their common intersection, draw CD at right angles to the opposite side ASB; from D, draw DA and DB at right angles to the edges SA and SB; join CA and CB.

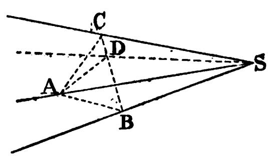
The straight lines CA and CB are at right angles to SA and SB; consequently the angles CAD and CBD are the measures of the dihedral angles A and B of the given trihedral.

But from the hypothesis and the construction, it follows that CA and CB, CAD and CBD, are equal. Therefore, the dihedral angles A and B, of which CAD and CBD are the measures, are equal.

The converse can be proved in the same manner.

147. *If in an isosceles trihedral angle, a plane be drawn at right angles to the base through the opposite edge, this plane shall bisect the vertical dihedral, and also the base;* and *conversely.*

148. *In a trihedral angle the greater side is opposite to the greater angle,* and *conversely.*



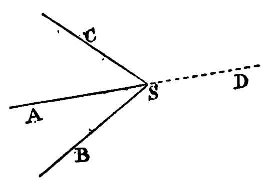

In the trihedral SABC, let the dihedral angle A be greater than the dihedral angle B.

Let the plane DSA, passing through SA, make with the plane ASB a dihedral angle equal to B, and let SD be its intersection with the plane BSC.

Then ASD and DSB are equal; to both add CSD, and ASD together with CSD shall be equal to BSC. But ASD and CSD are together greater than ASC. Therefore, BSC is greater than ASC.

The converse is easily proved by an indirect demonstration.

149. *In a trihedral angle, the sum of two dihedral angles and the sum of the two sides opposite to them are both greater than, equal to, or less than, two right angles.*




In the trihedral SABC, if the sides BSC and ASC are together greater than two right angles, the dihedral angles A and B opposite to them shall also be together greater than two right angles.

Produce AS to D; the three edges SD, SC, SB, will form a second trihedral in which the side BSC is greater than CSD, because ASC and CSD are together equal to two right angles, whereas ASC and BSC are greater than two right angles. Therefore, in the trihedral angle SBCD, the dihedral whose edge is SD or SA, shall be greater than the dihedral whose edge is SB. Consequently, the dihedrals A and B in the given trihedral SABC, shall be greater than

the two dihedrals formed by the plane BSC and the plane ASB produced, that is to say, greater than two right angles.

The other two parts of the proposition can be proved in the same way.

From this it follows that half the sum of the two dihedral angles of a trihedral, and half the sum of the sides opposite to them, are of *like affection.**

150. *In a right-angled trihedral an oblique angle and the side opposite to it are of like affection.*

151. *If from a point within a dihedral angle perpendiculars be drawn to the two sides, the angle contained by these perpendiculars shall be the supplement of the measure of the dihedral angle.*

152. *If from a point within a trihedral angle perpendiculars be drawn to the three sides, they shall be the edges of a second trihedral angle, whose sides and dihedrals are the supplements, respectively, of the dihedrals and sides of the first trihedral angle.*

Two trihedral angles so related to one another are said to be *supplementary.*

153. *The three dihedral angles of a trihedral are together less than six right angles, but greater than two.*

Let A, B, C, be the three dihedral angles of a trihedral, and $a'$, $b'$, $c'$, the corresponding sides of the supplementary trihedral.

Then $A=180°-a'$, $B=180°-b'$, $C=180°-c'$,

and $$A+B+C=3.180°-(a'+b'+c').$$

Hence, the sum of the three dihedrals is less than 3.180°, or six right angles.

Also the sum of the three sides $a'$, $b'$, $c'$, is less than four right angles, or 2.180°; therefore, the sum of the three dihedrals is greater than 180°, or two right angles.

154. The excess of the sum of the dihedrals of a trihedral angle above two right angles is called the *Spherical Excess*, and is usually represented by $\varepsilon$.

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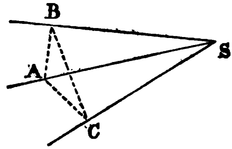
* A dihedral and a plane angle are said to be *alike*, *of like affection*, or *of the same kind*, when they are both acute or both obtuse.

155. *The excess of the sum of two dihedral angles of a trihedral above the third is always less than two right angles.*

156. *In a right-angled trihedral the cosine of the hypotenuse is equal to the product of the cosine of the two sides containing the right angle.*




Let SABC be the given trihedral angle in which the dihedral C is a right angle, and consequently the two planes ASC and BSC are perpendicular.

From any point B in the edge SB, draw BC at right angles to SC; from C draw CA at right angles to SA, and join AB.

Then BA is perpendicular to SA, and, consequently, the three triangles ASB, BSC, CSA, are right-angled in A, C, and A, respectively.

Therefore,

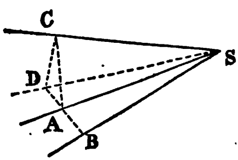
$$\cos ASB = \frac{SA}{SB}, \cos BSC = \frac{SC}{SB}, \cos ASC = \frac{SA}{SC};$$

whence $$\cos ASB = \cos BSC \cos ASC.$$

From this it will appear that in a right-angled trihedra the three sides must be acute, or two of them obtuse, the third being acute.

Compare this proposition with the 47th of the First Book.

157. *In an obtuse-angled trihedral the cosine of the side opposite to the obtuse angle is less than the product of the cosines of the two sides containing it by the continued product of the cosine of one of these sides, the sine of the other, and the tangent of the continuation of this last side to meet a plane drawn at right angles to it through the opposite edge.*




Let SABC be the given trihedral, having the dihedral angle A obtuse.

Through the edge SC let the plane CSD be drawn at right angles to the side ASB, meeting the continuation of this plane in SD.

Then, from the right-angled trihedral SDBC, by (156)

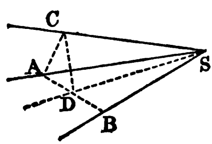
$$\cos CSB = \cos CSD \cos DSB$$
$$= \cos CSD \cos (DSA + ASB)$$
$$= \cos CSD \cos DSA \cos ASB - \cos CSD \sin DSA \sin ASB$$
$$= \cos CSA \cos ASB - \cos CSA \sin ASB \tan DSA,$$

because in the right-angled trihedral SCDA,

(156) $\cos CSD \cos DSA = \cos CSA$, and $\cos CSD = \dfrac{\cos CSA}{\cos DSA}$.

Compare this proposition with the 12th of the Second Book.

158. *In a trihedral the cosine of the side opposite to an acute dihedral angle exceeds the product of the cosines of the sides containing it by the continued product of the cosine of one of these sides, the sine of the other, and the tangent of the part of this latter side intercepted between the acute angle and a plane drawn at right angles to the side through the opposite edge.*




Let SABC be a trihedral, the dihedral angle A being acute; and let CSD be drawn through the edge SC at right angles to the opposite side ASB, meeting this side in SD.

Then, from the right-angled trihedral SDBC,

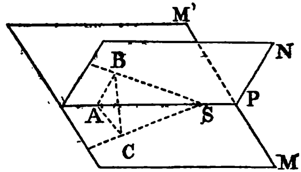
(156) $\cos CSB = \cos CSD \cos DSB$
$$= \cos CSD \cos (ASB - ASD)$$
$$= \cos CSD \cos ASB \cos ASD + \cos CSD \sin ASB \sin ASD$$
$$= \cos ASC \cos ASB + \cos ASC \sin ASB \tan ASD;$$

because, in the right-angled trihedral SCDA,

(156) $\cos CSD \cos DSA = \cos ASC$, and $\cos CSD = \dfrac{\cos ASC}{\cos ASD}$.

Compare this proposition with the 13th of the Second Book.

## ANGULAR FUNCTIONS OF DIHEDRAL ANGLES.




159. If the plane AN be conceived to turn on the straight line AP as a hinge, from the position of coincidence with the plane AM until it coincides with the continuation AM′ of this plane, the inclinations of the two planes during this revolution will determine so many dihedral angles.

And if a plane be drawn through any point of the edge at right angles to it, the angle formed by its intersections with the initial and the revolving planes will be the measures of these dihedral angles.

From any point B in the revolving plane AN, draw BC at right angles to the initial plane. Join any point S of the edge AP of the dihedral angle with B and C; the plane BSC is at right angles to the initial plane AM. From C draw CA at right angles to the edge AS, and join BA; the plane BAC is at right angles to the edge AS, and the angle BAC is the measure of the dihedral angle formed by the planes AM and AN. Let this dihedral be represented by A. Then,

$$\sin A = \sin BAC = \frac{BC}{BA} = \frac{BC}{BS} \div \frac{BA}{BS} = \frac{\sin BSC}{\sin BSA}$$

$$\tan A = \tan BAC = \frac{BC}{AC} = \frac{BC}{CS} \div \frac{AC}{CS} = \frac{\tan BSC}{\sin ASC}$$

$$\cos A = \cos BAC = \frac{AC}{AB} = \frac{AC}{AS} \div \frac{AB}{AS} = \frac{\tan ASC}{\tan ASB}.$$

160. If in one side of a dihedral angle a straight line be drawn meeting the edge, and through this straight line a plane be drawn at right angles to the other side,

The ratio of the sine of the angle in the perpendicular plane to the sine of the angle in the first mentioned plane is the *sine of the dihedral angle;*

The ratio of the tangent of the angle in the perpendicular plane to the sine of the angle in the side to which the perpendicular is drawn is the *tangent of the dihedral angle;*

The ratio of the tangent of the angle in the side to which the perpendicular is drawn to the tangent of the angle in the other side is the *cosine of the dihedral angle.*

The reciprocals of these angular functions will be the cosecant, cotangent, and secant of the dihedral.

161. The analogy between the antecedents and consequents of the ratios which form the angular functions of a dihedral, and the antecedents and consequents of the corresponding functions of a plane angle, is evident.

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## LESSON XVI.

Formulæ for the solution of Right-Angled Trihedrals—Napier's Rule of Circular Parts.

### FORMULÆ FOR THE SOLUTION OF RIGHT-ANGLED TRIHEDRALS.

162. In the solution of right-angled trihedrals five cases may be distinguished. There may be given—

1. The two sides containing the right angle,
2. One side and an angle,
3. The hypotenuse and an angle,
4. The hypotenuse and a side,
5. The two angles;

and it may be required to determine the three remaining parts, or any one of them.

The right angle will be represented by C.

163. CASE I.—*Given the two sides containing the right angle, to find the other parts of the trihedral.*

Then (160) $\tan A = \frac{\tan a}{\sin b}$, $\tan B = \frac{\tan b}{\sin a}$,

and (156) $\cos c = \cos a \cos b$.

The logarithmic formulæ are,

$$L \tan A = L \tan a + cL \sin b, \quad L \tan B = L \tan b + cL \sin a$$

$$L \cos c = L \cos a + L \cos b - 10.$$

164. CASE II.—*Given one side and an angle, to find the remaining parts of the right-angled trihedral.*

I. Let A be the given angle, and $a$ the given side.

Then (160) $\sin A=\frac{\sin a}{\sin c}$, $\tan A=\frac{\tan a}{\sin b}$;

whence $\sin c=\frac{\sin a}{\sin A}$, $\sin b=\frac{\tan a}{\tan A}$.

By (160) $\sin B=\frac{\sin b}{\sin c}$, and by substitution,

$$\sin B=\frac{\tan a \sin A}{\sin a \tan A}=\frac{\cos A}{\cos a}.$$

Each of the three required parts, being determined by the sine, will have two values; but these values cannot be taken indiscriminately the one with the other.

If the given side $a$ be acute, with the acute value of $c$ must be taken (156, 150) the acute values of $b$ and B; and with the obtuse value of $c$ must be taken the obtuse values of $b$ and B.

If, on the other hand, $a$ be obtuse, with the acute value of $c$ must be taken the obtuse values of $b$ and B; and with the obtuse value of $c$ must be taken the acute values of $b$ and B.

Besides, that the problem may be possible, if $a$ be acute, A must also be acute, and its measure greater than that of $a$; and if $a$ be obtuse, A must be obtuse, and its measure less than that of $a$.

The problem, when possible, admits of two solutions. But there is no ambiguity in the case, although it is sometimes, though not properly, referred to as an ambiguous case.

Expressed logarithmically, the formulæ will be as follows:

$$\text{L}\sin c=\text{L}\sin a+c\text{L}\sin A,\ \text{L}\sin b=\text{L}\tan a+c\text{L}\tan A$$
$$\text{L}\sin B=\text{L}\cos A+c\text{L}\cos a.$$

II. Let A be the given angle, and $b$ the given side.

From (160) $\frac{\tan b}{\tan c}=\cos A$, $\frac{\tan a}{\sin b}=\tan A$

whence $\tan c=\frac{\tan b}{\cos A}$, $\tan a=\sin b \tan A$.

To determine B, the definition (160) gives

$$\cos B=\frac{\tan a}{\tan c},$$

and by substitution,

$$\cos B=\frac{\sin b \tan A \cos A}{\tan b}=\cos b \sin A.$$

The logarithmic formulæ are

$$\text{L}\tan c=\text{L}\tan b+c\text{L}\cos A,\ \text{L}\tan a=\text{L}\sin b+\text{L}\tan A-10$$
$$\text{L}\cos B=\text{L}\cos b+\text{L}\sin A-10.$$

165. Case III.—*Given the hypotenuse and an angle, to find the other parts of the trihedral.*

Let A be the given angle.

By (160) $\frac{\sin a}{\sin c}=\sin A, \frac{\tan b}{\tan c}=\cos A,$

whence $\sin a=\sin c \sin A, \tan b=\tan c \cos A.$

The angle B will be obtained from the expression $\tan B=\frac{\tan b}{\sin a}$, by substituting for $\tan b$ and $\sin a$ their values in function of $c$ and A. This will give

$$\tan B=\frac{\tan c \cos A}{\sin c \sin A}=\frac{\cot A}{\cos c}.$$

Whence

$$L \sin a=L \sin c+L \sin A-10, L \tan b=L \tan c+L \cos A-10$$
$$L \tan B=L \cot A+cL \cos c.$$

166. Case IV.—*Given in a right-angled trihedral the hypotenuse and a side, to find the remaining parts.*

Let $a$ be the given side.

Then (160) $\sin A=\frac{\sin a}{\sin c}, \cos B=\frac{\tan a}{\tan c},$

and (156) $\cos c=\cos a \cos b$, whence $\cos b=\frac{\cos c}{\cos a}.$

Although the angle A is determined by its sine, the problem admits but one solution, because the angle A is of like affection with the given side $a$.

The logarithmic expressions are

$$L \sin A=L \sin a+cL \sin c, \ L \cos B=L \tan a+cL \tan c$$
$$L \cos b=L \cos c+cL \cos a.$$

Case V.—*Given the two oblique angles of a right-angled trihedral, to find the other parts.*

From the values for the sine, cosine, and tangent, of an angle, as given in (160), will be obtained

$$\frac{\cos A}{\sin B}=\frac{\tan b \sin c}{\tan c \sin b}=\frac{\cos c}{\cos b}=\cos a$$
$$\frac{\cos B}{\sin A}=\frac{\tan a \sin c}{\tan c \sin a}=\frac{\cos c}{\cos a}=\cos b$$
$$\cot A \cot B=\frac{\sin a \sin b}{\tan a \tan b}=\cos a \cos b=\cos c.$$

Expressed logarithmically, these formulæ become

$$\text{L}\cos a = \text{L}\cos \text{A} + c\text{L}\sin \text{B},\quad \text{L}\cos b = \text{L}\cos \text{B} + c\,\text{L}\sin \text{A}$$

$$\text{L}\cos c = \text{L}\cot \text{A} + \text{L}\cot \text{B} - 10.$$

167. Compare the formulæ in the first four cases with those given for the solution of right-angled triangles.

### NAPIER'S RULE FOR CIRCULAR PARTS.

168. Let the two sides containing the right angle of a right-angled trihedral angle, the complement of the hypotenuse, and the complements of the oblique angles, that is,

$$a,\ 90^\circ - \text{B},\ 90^\circ - c,\ 90^\circ - \text{A},\ b,$$

be called the five *circular parts* of the trihedral. With reference to any one of these circular parts, two of the other four are said to be *adjacent*, and two to be *opposite* or *remote*. For instance, with reference to $90^\circ - c$, the two circular parts $90^\circ - \text{A}$, and $90^\circ - \text{B}$ are adjacent, and $a$ and $b$ opposite. With reference to the circular part $a$, $b$ and $90^\circ - \text{B}$ are adjacent, $90^\circ - c$ and $90^\circ - \text{A}$ are opposite.

All the formulæ for the solution of the five foregoing cases of right-angled trihedrals are included in the following rule, which is called *Napier's Rule of circular parts*:—

*The sine of a circular part is equal to the product of the tangents of the two adjacent circular parts, or to the product of the cosines of the opposite circular parts.*

Suppose $a$ and $b$ given, as in Case I.

I. To determine $c$.

Considering the relative position of the circular parts $a$, $b$, and $90^\circ - c$, it will appear that with reference to $90^\circ - c$, the circular parts $a$ and $b$ are opposite.

Therefore, $\sin(90^\circ - c) = \cos a \cos b$, or $\cos c = \cos a \cos b$.

II. To determine A.

If the relative position of the circular parts $90^\circ - \text{A}$, $a$ and $b$, be examined, it will appear that $90^\circ - \text{A}$ and $a$ are adjacent with reference to $b$.

Hence, $\sin b = \tan a \tan(90^\circ - \text{A})$, or $\sin b = \tan a \cot \text{A}$.

III. To determine B.

A similar consideration of the relative position of the circular parts $90^\circ - \text{B}$, $a$ and $b$ will give

$$\sin a = \tan b \tan(90^\circ - \text{B}),\ \text{or}\ \sin a = \tan b \cot \text{B}.$$

Suppose $a$ and A given.

I. To find $c$.

With reference to the circular part $a$, the circular parts $90°—A$ and $90°—c$ will be found to be opposite. Whence,

$$\sin a = \cos(90° - c)\cos(90° - A), \text{ or } \sin a = \sin c \sin A.$$

II. To determine $b$.

It will be seen that $90°—A$ and $a$ are adjacent with reference to $b$, and, therefore, by Napier's Rule,

$$\sin b = \tan a \tan(90° - A), \text{ or } \sin b = \tan a \cot A.$$

III. To determine B.

The circular parts $a$ and $90°—B$ are opposite with reference to $90°—A$, and, therefore,

$$\sin(90° - A) = \cos a \cos(90° - B), \text{ or } \cos A = \cos a \sin B.$$

These formulæ correspond with those obtained in Case II.

All the other formulæ will be got by a similar process.

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## LESSON XVII.

**Relations connecting the angular functions of the sides and angles of a trihedral.**

169. *In a trihedral angle the sines of the dihedral angles and of the sides opposite to them are proportional.*

Let SABC [see diagrams (157 and 158)] be a trihedral, and from the edge SC let the plane CSD be drawn at right angles to the opposite side ASB.

Then (160) $\sin A = \dfrac{\sin CSD}{\sin CSA}$, $\sin B = \dfrac{\sin CSD}{\sin CSB}$,

whence, $$\frac{\sin A}{\sin B} = \frac{\sin CSB}{\sin CSA} = \frac{\sin a}{\sin b}.$$

In will be proved in like manner that,

$$\frac{\sin A}{\sin C} = \frac{\sin a}{\sin c}, \text{ and } \frac{\sin B}{\sin C} = \frac{\sin b}{\sin c}.$$

Therefore,

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

170. *In a trihedral angle the cosine of a dihedral angle is equal to the quotient obtained by dividing by the product of the sines of the two sides containing the angle, the excess of the cosine of the third side over the product of the cosines of the first mentioned sides.*

From (158)

$$\cos a = \cos b \cos c + \cos b \sin c \tan \text{ASD}$$

whence,

$$\tan \text{ASD} = \frac{\cos a - \cos b \cos c}{\cos b \sin c}$$

$$\frac{\tan \text{ASD}}{\tan \text{ASC}} = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

But

$$\frac{\tan \text{ASD}}{\tan \text{ASC}} = \cos \text{A}.$$

Again from (157)

$$\cos a = \cos b \cos c - \cos b \sin c \tan \text{ASD}$$

$$-\tan \text{ASD} = \frac{\cos a - \cos b \cos c}{\cos b \sin c}$$

$$-\frac{\tan \text{ASD}}{\tan \text{ASC}} = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

And

$$-\frac{\tan \text{ASD}}{\tan \text{ASC}} = -\cos \text{CSASD}^* = \cos \text{A}.$$

Therefore, whether the dihedral angle be acute or obtuse,

$$\cos \text{A} = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

Compare this theorem and the preceding one with those in (45) and (46).

171. *In a trihedral angle the cosine of one of the sides is equal to the quotient obtained by dividing by the product of the sines of the dihedral angles adjacent to it, the sum of the cosine of the third dihedral angle and the product of the cosines of the first-mentioned dihedrals.*

Let $\text{A}'$, $\text{B}'$, $\text{C}'$, $a'$, $b'$, $c'$, be the dihedrals and sides of the supplementary trihedral.

By (170)

$$\cos \text{A}' = \frac{\cos a' - \cos b' \cos c'}{\sin b' \sin c'}.$$

* Five letters thus arranged represent a dihedral angle formed by two plane angles, CSA and ASD, whose common intersection is SA.

But $A'=180°-a$, $a'=180°-A$, $b'=180°-B$, $c'=180°-C$; therefore, $\cos(180°-a)=$

$$\frac{\cos(180°-A)-\cos(180°-B)\cos(180°-C)}{\sin(180°-B)\sin(180°-C)},$$

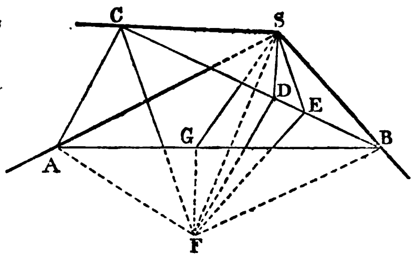
or,

$$\cos a=\frac{\cos A+\cos B\cos C}{\sin B\sin C}.$$

172. *To prove Napier's Analogies:*

$$\frac{\tan\frac{1}{2}(A+B)}{\cot\frac{1}{2}C}=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)},\quad \frac{\tan\frac{1}{2}(A-B)}{\cot\frac{1}{2}C}=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}$$

$$\frac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}c}=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)},\quad \frac{\tan\frac{1}{2}(a-b)}{\tan\frac{1}{2}c}=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}$$




Let SABC be the given trihedral, the dihedral A being greater than B.

Make CSD equal to CSA; and bisect the angle DSB by the straight line SE. Through this line SE draw a plane at right angles to CSB; and through the edge SC a plane bisecting the dihedral C, and meeting in SF the plane at right angles to CSB. Through SF draw a plane at right angles to ASB and meeting it in SG.

By construction, the sides ASF, DSF, BSF, are equal to one another; and

$$CSE=\tfrac{1}{2}(a+b),\ ESB=\tfrac{1}{2}(a-b).$$

The dihedrals CSASF and CSDSF are equal, because the dihedrals CSASD and DSASF, of which the former is made up, are respectively equal to the dihedrals CSDSA and ASDSF, of which the latter is made up.

The dihedrals BSASF and ASBSF, FSDSB and FSBSD, are equal (147).

Also, $GSFSB=\tfrac{1}{2}ASFSB=\tfrac{1}{2}(ASFSD+DSFSB)$
$=\tfrac{1}{2}(2.CSFSD+2.DSFSE)$
$=ESFSC,$

Now, A=CSASF—BSASF
=CSDSF—ASBSF
=180°—FSDSB—ASBSF
=180°—FSBSD—ASBSF
and B=FSBSD—ASBSF

Hence, $\frac{1}{2}(A+B)=90°$—ASBSF, $\frac{1}{2}(A-B)=90°$—FSBSD.

But the right-angled trihedrals SBGF and SBEF give

cot ASBSF cot GSFSB=cos BSF=cos BSE cos ESF,

and the trihedral SCEF gives,

cot ESCSF cot ESFSC=cos CSF=cos CSE cos ESF.

And by division,

$$\frac{\text{cot ASBSF}}{\text{cot ESCSF}}=\frac{\text{cos BSE}}{\text{cos CSE}}, \text{ that is, } \frac{\tan\frac{1}{2}(A+B)}{\cot\frac{1}{2}C}=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}.$$

The trihedrals SBEF and SCEF give

cot FSBSE tan ESF=sin BSE,
cot ESCSF tan ESF=sin CSE;

whence, by division,

$$\frac{\text{cot FSBSE}}{\text{cot ESCSF}}=\frac{\text{sin BSE}}{\text{sin CSE}}, \text{ that is, } \frac{\tan\frac{1}{2}(A-B)}{\cot\frac{1}{2}C}=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}.$$

The trihedrals SCEF, SBFG, SBEF, give (the dihedrals CSFSE and GSFSB being equal),

tan CSE=sin ESF tan GSFSB,
tan BSG=sin BSF sin GSBSF tan GSFSB,

and by division,

$$\frac{\text{tan CSE}}{\text{tan BSG}}=\frac{\text{sin CSBSF}}{\text{sin GSBSF}}, \text{ that is, } \frac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}c}=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}.$$

Lastly the trihedrals SBFG, SBEF, give

tan BSE=tan BSF cos CSBSF,
tan BSG=tan BSF cos ASBSF,

whence

$$\frac{\text{tan BSE}}{\text{tan BSG}}=\frac{\text{cos CSBSF}}{\text{cos ASBSF}}, \text{ that is, } \frac{\tan\frac{1}{2}(a-b)}{\tan\frac{1}{2}c}=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}.$$

173. These analogies may be investigated analytically as follows.

The theorems (169, 170, 171) give

$$\frac{\sin A \sin B}{\sin^2 C}=\frac{\sin a \sin b}{\sin^2 c}$$

$$\cos a \cos b=\cos c-\sin a \sin b \cos C$$

$$\cos A \cos B=-\cos C+\sin A \sin B \cos c.$$

Then

$$\frac{\tan^2 \frac{1}{2}(A+B)}{\cot^2 \frac{1}{2}C}=\frac{1-\cos(A+B)}{1+\cos(A+B)}\times\frac{1-\cos C}{1+\cos C}$$

$$=\frac{1-\cos A \cos B+\sin A \sin B}{1+\cos A \cos B-\sin A \sin B}\times\frac{1-\cos C}{1+\cos C}$$

$$=\frac{1+\cos C+\sin A \sin B(1-\cos c)}{1-\cos C-\sin A \sin B(1-\cos c)}\times\frac{1-\cos C}{1+\cos C}$$

$$=\frac{\sin^2 C+\sin A \sin B(1-\cos c)(1-\cos C)}{\sin^2 C-\sin A \sin B(1-\cos c)(1+\cos C)}$$

$$=\frac{1+\dfrac{\sin A \sin B}{\sin^2 C}(1-\cos c)(1-\cos C)}{1-\dfrac{\sin A \sin B}{\sin^2 C}(1-\cos c)(1+\cos C)}$$

$$=\frac{1+\dfrac{\sin a \sin b}{\sin^2 c}(1-\cos c)(1-\cos C)}{1-\dfrac{\sin a \sin b}{\sin^2 c}(1-\cos c)(1+\cos C)}$$

$$=\frac{\sin^2 c+\sin a \sin b(1-\cos c)(1-\cos C)}{\sin^2 c-\sin a \sin b(1-\cos c)(1+\cos C)}$$

$$=\frac{1+\cos c+\sin a \sin b(1-\cos C)}{1+\cos c-\sin a \sin b(1+\cos C)}$$

$$=\frac{1+\sin a \sin b+\cos a \cos b}{1+\cos a \cos b-\sin a \sin b}$$

$$=\frac{1+\cos(a-b)}{1+\cos(a+b)}$$

$$=\frac{\cos^2 \frac{1}{2}(a-b)}{\cos^2 \frac{1}{2}(a+b)}.$$

Whence

$$\frac{\tan \frac{1}{2}(A+B)}{\cot \frac{1}{2}C}=\frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)}.$$

The other three analogies may be obtained by a similar process.

The extraction of the square root will not lead to any ambiguity; for the angles $\frac{1}{2}C$, $\frac{1}{2}c$, $\frac{1}{2}(A-B)$, and $\frac{1}{2}(a-b)$, being acute, their angular functions will all be positive; and $\frac{1}{2}(A+B)$ and $\frac{1}{2}(a+b)$ being of like affection (149), the tan-

gent of the one and the cosine of the other will have the same sign.

Compare these formulæ with that in (47).

174. *To prove Delambre's, or Gauss', formulæ:**

$$\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c}=\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}, \quad \frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}=\frac{\cos\frac{1}{2}(A+B)}{\sin\frac{1}{2}C}$$

$$\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}c}=\frac{\sin\frac{1}{2}(A-b)}{\cos\frac{1}{2}C}, \quad \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}=\frac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}C}$$

These formulæ may be derived from the same construction as the analogies, or investigated analytically after the same manner.

For instance,

$$\frac{\sin^2\frac{1}{2}(a+b)}{\sin^2\frac{1}{2}c}=\frac{1-\cos(a+b)}{1-\cos c}$$

$$=\frac{1-\cos a\cos b+\sin a\sin b}{1-\cos c}$$

$$=\frac{1-\cos c+\sin a\sin b(1+\cos C)}{1-\cos c}$$

$$=1+\frac{\sin a\sin b}{\sin^2 c}(1+\cos C)(1+\cos c)$$

$$=1+\frac{\sin A\sin B}{\sin^2 C}(1+\cos C)(1+\cos c)$$

$$=\frac{1-\cos C+\sin A\sin B\cos c+\sin A\sin B}{1-\cos C}$$

$$=\frac{1+\cos A\cos B+\sin A\sin B}{1-\cos C}$$

$$=\frac{1+\cos(A-B)}{1-\cos C}$$

$$=\frac{\cos^2\frac{1}{2}(A-B)}{\sin^2\frac{1}{2}C}$$

Therefore,

$$\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c}=\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}.$$

The others can be proved in the same way.

Compare these formulæ with those in (48) and (49).

175. Napier's analogies can be at once derived from Delambre's formulæ by division.

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* Attributed by most English writers to Gauss, who published them in 1809, but given two years earlier by Delambre.

# LESSON XVIII.

**Formulæ for the solution of Oblique-Angled Trihedrals.**

189. There are five cases to be considered in the solution of oblique-angled trihedrals. There may be given—

1. Two sides and the contained angle,
2. Three sides,
3. A side and two angles,
4. Two sides, and the angle opposite to one of them,
5. Three angles.

190. CASE I.—*Given the two sides of a trihedral angle and the dihedral angle they contain, to find the remaining parts.*

Let $a$ and $b$ be the given sides, $a$ being the greater of the two, and C the contained angle.

The analogies give,

$$\tan \tfrac{1}{2}(A+B)=\cot \tfrac{1}{2}C\frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)},$$

$$\tan \tfrac{1}{2}(A-B)=\cot \tfrac{1}{2}C\frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)},$$

$$\tan \tfrac{1}{2}c=\tan \tfrac{1}{2}(a+b)\frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)}.$$

Expressed logarithmically, the formulæ become

$$(1)\quad \begin{cases} L\tan \frac{1}{2}(A+B)=L\cot \frac{1}{2}C+L\cos \frac{1}{2}(a-b) \\ \qquad\qquad +cL\cos \frac{1}{2}(a+b)-10 \\ L\tan \frac{1}{2}(A-B)=L\cot \frac{1}{2}C+L\sin \frac{1}{2}(a-b) \\ \qquad\qquad +cL\sin \frac{1}{2}(a+b)-10 \\ L\tan \frac{1}{2}c=L\tan \frac{1}{2}(a+b)+L\cos \frac{1}{2}(A+B) \\ \qquad\qquad +cL\cos \frac{1}{2}(A-B)-10, \end{cases}$$

requiring in all six openings of the tables.

If the two given sides, $a$ and $b$, be such that their sum does not exceed 180°, these logarithmic expressions will solve the problem.

If, on the other hand, the sum of the two given sides $a$ and $b$ exceeds 180°, the formulæ have to be modified for the logarithmic computation.

When $a+b$ is greater than $180°$, $\frac{1}{2}(a+b-180°)$ will be an acute angle, and so will $\frac{1}{2}(A+B-180°)$. Then,

$$\begin{aligned}
\sin\tfrac{1}{2}(a+b)&=\cos\tfrac{1}{2}(a+b-180°)\\
\cos\tfrac{1}{2}(a+b)&=-\sin\tfrac{1}{2}(a+b-180°)\\
\tan\tfrac{1}{2}(a+b)&=-\cot\tfrac{1}{2}(a+b-180°)\\
\cos\tfrac{1}{2}(A+B)&=-\sin\tfrac{1}{2}(A+B-180°)\\
\tan\tfrac{1}{2}(A+B)&=-\cot\tfrac{1}{2}(A+B-180°).
\end{aligned}$$

Substituting these in the formulæ, the logarithmic expressions will be,

$$(2)\begin{cases}
L\cot\tfrac{1}{2}(A+B-180°)=L\cot\tfrac{1}{2}C+L\cos\tfrac{1}{2}(a-b)\\
\qquad\qquad +cL\sin\tfrac{1}{2}(a+b-180°)-10\\
L\tan\tfrac{1}{2}(A-B)=L\cot\tfrac{1}{2}C+L\sin\tfrac{1}{2}(a-b)\\
\qquad\qquad +cL\cos\tfrac{1}{2}(a+b-180°)-10\\
L\tan\tfrac{1}{2}c=L\cot\tfrac{1}{2}(a+b-180°)+L\sin\tfrac{1}{2}(A+B-180°)\\
\qquad\qquad +cL\cos\tfrac{1}{2}(A-B)-10
\end{cases}$$

191. CASE II.—*Given the three sides of a trihedral, to find the angles.*

The theorem (170) gives

$$\cos A=\frac{\cos a-\cos b\cos c}{\sin b\sin c},\ \cos B=\frac{\cos b-\cos a\cos c}{\sin a\sin c},$$

$$\cos C=\frac{\cos c-\cos a\cos b}{\sin a\sin b};$$

and these formulæ are sufficient for the solution of the problem. But for logarithmic computation others have to be substituted for them.

The value of $\tan^2\frac{1}{2}A$, in function of $\cos A$ (52), becomes after substitution and multiplication of both terms of the fraction by $\sin b\sin c$,

$$\begin{aligned}
\tan^2\tfrac{1}{2}A&=\frac{\sin b\sin c-\cos a+\cos b\cos c}{\sin b\sin c+\cos a-\cos b\cos c}\\
&=\frac{\cos(b-c)-\cos a}{\cos a-\cos(b+c)}\\
&=\frac{\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(a+c-b)}{\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(b+c-a)}
\end{aligned}$$

If the sum of the three sides be represented by $2s$, then

$$\begin{aligned}
a+b+c&=2s & a+c-b&=2s-2b\\
a+b-c&=2s-2c & b+c-a&=2s-2a.
\end{aligned}$$

Therefore,

$$\tan\tfrac{1}{2}A=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}.$$

The positive value alone of the root is to be taken, because the half of an angle of a trihedral is always acute.

The values for tan $\frac{1}{2}$B, tan $\frac{1}{2}$C, may be obtained in the same way.

The logarithmic form will be

$$\text{L}\tan\tfrac{1}{2}\text{A}=\tfrac{1}{2}\left\{\begin{array}{l}\text{L}\sin(s-b)+\text{L}\sin(s-c)\\ +c\text{L}\sin s+c\text{L}\sin(s-a)\end{array}\right\}$$

The student will have no difficulty in verifying the expressions

$$\sin\tfrac{1}{2}\text{A}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}},$$

$$\cos\tfrac{1}{2}\text{A}=\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}},$$

$$\sin\text{A}=\frac{2}{\sin b\sin c}\sqrt{\sin s\sin(s-a)\sin(s-b)\sin(s-c)}.$$

192. CASE III.—*Given a side and two angles of a trihedral, to find the remaining sides and angle.*

I. Let the given side be adjacent to the two given angles. For instance, let $c$, A, B, be given; A being greater than B.

From Napier's analogies,

$$\tan\tfrac{1}{2}(a+b)=\tan\tfrac{1}{2}c\frac{\cos\frac{1}{2}(\text{A}-\text{B})}{\cos\frac{1}{2}(\text{A}+\text{B})},$$

$$\tan\tfrac{1}{2}(a-b)=\tan\tfrac{1}{2}c\frac{\sin\frac{1}{2}(\text{A}-\text{B})}{\sin\frac{1}{2}(\text{A}+\text{B})},$$

$$\cot\tfrac{1}{2}\text{C}=\tan\tfrac{1}{2}(\text{A}+\text{B})\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}.$$

The logarithmic expressions will be, when A+B does not exceed 180°,

$$(1)\left\{\begin{array}{l}\text{L}\tan\tfrac{1}{2}(a+b)=\text{L}\tan\tfrac{1}{2}c+\text{L}\cos\tfrac{1}{2}(\text{A}-\text{B})\\ \qquad\qquad +c\text{L}\cos\tfrac{1}{2}(\text{A}+\text{B})-10\\ \text{L}\tan\tfrac{1}{2}(a-b)=\text{L}\tan\tfrac{1}{2}c+\text{L}\sin\tfrac{1}{2}(\text{A}-\text{B})\\ \qquad\qquad +c\text{L}\sin\tfrac{1}{2}(\text{A}+\text{B})-10\\ \text{L}\cot\tfrac{1}{2}\text{C}=\text{L}\tan\tfrac{1}{2}(\text{A}+\text{B})+\text{L}\cos\tfrac{1}{2}(a+b)\\ \qquad\qquad +c\text{L}\cos\tfrac{1}{2}(a-b)-10,\end{array}\right.$$

and when A+B exceeds 180°,

$$(2)\left\{\begin{array}{l}\text{L}\cot\tfrac{1}{2}(a+b-180^\circ)=\text{L}\tan\tfrac{1}{2}c+\text{L}\cos\tfrac{1}{2}(\text{A}-\text{B})\\ \qquad\qquad +c\text{L}\sin\tfrac{1}{2}(\text{A}+\text{B}-180^\circ)-10\\ \text{L}\tan\tfrac{1}{2}(a-b)=\text{L}\tan\tfrac{1}{2}c+\text{L}\sin\tfrac{1}{2}(\text{A}-\text{B})\\ \qquad\qquad +c\text{L}\cos\tfrac{1}{2}(\text{A}+\text{B}-180^\circ)-10\\ \text{L}\cot\tfrac{1}{2}c=\text{L}\cot\tfrac{1}{2}(\text{A}+\text{B}-180^\circ)+\text{L}\cos\tfrac{1}{2}(a+b-180^\circ)\\ \qquad\qquad +c\text{L}\cos\tfrac{1}{2}(a-b)-10.\end{array}\right.$$

II. Let the given side be opposite to one of the given angles. For instance, let $a$, A, B, be given, A being the greater of the two angles.

From (169) $\sin b = \sin a \dfrac{\sin B}{\sin A}$,

and from (172) $\cot \frac{1}{2}C = \tan \frac{1}{2}(A+B) \dfrac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}(a-b)}$,

$$\tan \tfrac{1}{2}c = \tan \tfrac{1}{2}(a+b) \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)}.$$

As one of the required parts $b$ is obtained by its sine, it appears that the problem will, in some instances, admit of two solutions.

If A+B be less than 180°, and $A > B$, then the acute value of $b$ alone answers the question; for if $a$ be obtuse, as $a+b$ must be less than 180°, $b$ must be acute; and if $a$ be acute, as $a > b$, $b$ must also be acute.

If A+B be less than 180°, but $A < B$, there will be two solutions when $a$ is acute; and the problem will be impossible when $a$ is obtuse.

If A+B be equal to 180°, $b$ will be the supplement of $a$; and there will, therefore, be only one solution.

If A+B be greater than 180°, and $A < B$, there will be but one solution if $a$ be obtuse, and two if $a$ be acute.

If A+B be greater than 180°, but $A > B$, the two values of $b$ will answer if $a$ be obtuse, and the problem will be impossible if $a$ be acute.

The problem will further be impossible if $\sin a \sin B$ be greater than $\sin A$.

When written out for logarithmic computation, the formulæ will be

$$L \sin b = L \sin a + L \sin B + cL \sin A - 10;$$

and, when $a+b$ is less than 180°,

$$(1)\left\{\begin{aligned} L \cot \tfrac{1}{2}C &= L \tan \tfrac{1}{2}(A+B) + L \cos \tfrac{1}{2}(a+b) \\ &\quad + cL \cos \tfrac{1}{2}(a-b) - 10 \\ L \tan \tfrac{1}{2}c &= L \tan \tfrac{1}{2}(a+b) + L \cos \tfrac{1}{2}(A+B) \\ &\quad + cL \cos \tfrac{1}{2}(A-B) - 10, \end{aligned}\right.$$

when $a+b$ is greater than 180°,

$$(2)\left\{\begin{aligned} L \cot \tfrac{1}{2}C &= L \cot \tfrac{1}{2}(A+B-180°) + L \sin \tfrac{1}{2}(a+b-180°) \\ &\quad + cL \cos \tfrac{1}{2}(a-b) - 10 \\ L \tan \tfrac{1}{2}c &= L \cot \tfrac{1}{2}(a+b-180°) + L \sin \tfrac{1}{2}(A+B-180°) \\ &\quad + cL \cos \tfrac{1}{2}(A-B) - 10. \end{aligned}\right.$$

193. CASE IV.—*Given two sides of a trihedral and the angle opposite to one of them, to find the remaining parts.*

Let $a$ and $b$ be the given sides, and A the given angle.

From (169) $$\sin B = \frac{\sin b}{\sin a} \sin A$$

and from (172)

$$\cot \tfrac{1}{2}C = \tan \tfrac{1}{2}(A+B) \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}(a-b)},$$

$$\tan \tfrac{1}{2}c = \tan \tfrac{1}{2}(a+b) \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)}.$$

This problem, like the second part of the foregoing one, admits sometimes of two solutions.

If $a+b$ be less than 180°, and $a > b$, the acute value for B will alone satisfy the conditions.

If $a+b$ be less than 180°, but $a < b$, the angle B must be obtuse when A is obtuse; but when A is acute, both values of B may be taken.

If $a+b=180°$, there will be but one solution, because the angles A and B will then be supplements.

If $a+b$ be greater than 180°, and $a < b$, there will be one solution if A be obtuse, and two if A be acute.

If $a+b$ be greater than 180°, and $a > b$, the two values of B will answer the conditions, if A be obtuse; but if A be acute, the problem will be impossible.

It will also be impossible if $\sin b \sin A$ be greater than $\sin a$.

The logarithmic formulæ for this case will be

$$L \sin B = L \sin A + L \sin b + cL \sin a - 10,$$

together with the expressions [1] or [2], in the second part of the foregoing case according as A+B is less or greater than two right angles.

194. CASE V.—*Given the three angles of a trihedral angle, to find the sides.*

The formulæ for the solution are those established in theorem (171), namely,

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}, \quad \cos b = \frac{\cos B + \cos A \cos C}{\sin A \sin C},$$

$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B}.$$

Formulæ adapted to logarithmic computation may be obtained as in (191).

Thus,
$$\tan^2 \tfrac{1}{2}a = \frac{1-\cos a}{1+\cos a}$$
$$= \frac{\sin B \sin C - \cos A - \cos B \cos C}{\sin B \sin C + \cos A + \cos B \cos C}$$
$$= \frac{-[\cos (B+C) + \cos A]}{\cos (B-C) + \cos A}$$
$$= \frac{-\cos \tfrac{1}{2}(A+B+C) \cos \tfrac{1}{2}(B+C-A)}{\cos \tfrac{1}{2}(A+B-C) \cos \tfrac{1}{2}(A+C-B)}.$$

The angle $A+B+C$ being greater than two right angles, and less than six, the cosine of its half will be negative; and each of the other three angles $\tfrac{1}{2}(B+C-A)$, $\tfrac{1}{2}(A+B-C)$, $\tfrac{1}{2}(A+C-B)$, being acute (155), this value of $\tan^2 \tfrac{1}{2}a$ will be positive.

Hence, by making $A+B+C=2S$, and extracting the square root,

$$\tan \tfrac{1}{2}a = \sqrt{\frac{-\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}}.$$

Similarly,

$$\tan \tfrac{1}{2}b = \sqrt{\frac{-\cos S \cos (S-B)}{\cos (S-A) \cos (S-C)}},$$
$$\tan \tfrac{1}{2}c = \sqrt{\frac{-\cos S \cos (S-C)}{\cos (S-A) \cos (S-B)}}.$$

The following formulæ may also be found,

$$\sin \tfrac{1}{2}a = \sqrt{\frac{-\cos S \cos (S-A)}{\sin B \sin C}},$$
$$\cos \tfrac{1}{2}a = \sqrt{\frac{\cos (S-B) \cos (S-C)}{\sin B \sin C}},$$
$$\sin a = \frac{2\sqrt{-\cos S \cos (S-A) \cos (S-B) \cos (S-C)}}{\sin B \sin C}.$$

All the preceding values may be expressed in function of the spherical excess $\varepsilon$.

Because $A+B+C-180°=\varepsilon$, then

$\tfrac{1}{2}(A+B+C)=90°+\tfrac{1}{2}\varepsilon$, $\quad \tfrac{1}{2}(A+C-B)=90°+\tfrac{1}{2}\varepsilon-B$,
$\tfrac{1}{2}(A+B-C)=90°+\tfrac{1}{2}\varepsilon-C$, $\quad \tfrac{1}{2}(B+C-A)=90°+\tfrac{1}{2}\varepsilon-A$,
and

$\cos \tfrac{1}{2}(A+B+C)=-\sin \tfrac{1}{2}\varepsilon$, $\quad \cos \tfrac{1}{2}(A+C-B)=\sin (B-\tfrac{1}{2}\varepsilon)$,
$\cos \tfrac{1}{2}(A+B-C)=\sin (C-\tfrac{1}{2}\varepsilon)$, $\quad \cos \tfrac{1}{2}(B+C-A)=\sin (A-\tfrac{1}{2}\varepsilon)$.

Therefore,

$$\tan \tfrac{1}{2}a = \sqrt{\frac{\sin \frac{1}{2}\epsilon \sin (A-\frac{1}{2}\epsilon)}{\sin (B-\frac{1}{2}\epsilon) \sin (C-\frac{1}{2}\epsilon)}},$$

$$\sin \tfrac{1}{2}a = \sqrt{\frac{\sin \frac{1}{2}\epsilon \sin (A-\frac{1}{2}\epsilon)}{\sin B \sin C}},$$

$$\cos \tfrac{1}{2}a = \sqrt{\frac{\sin (B-\frac{1}{2}\epsilon) \sin (C-\frac{1}{2}\epsilon)}{\sin B \sin C}},$$

$$\sin a = \frac{2}{\sin B \sin C} \sqrt{\sin \tfrac{1}{2}\epsilon \sin (A-\tfrac{1}{2}\epsilon) \sin (B-\tfrac{1}{2}\epsilon) \sin (C-\tfrac{1}{2}\epsilon)}.$$

197. The first, third, and fourth cases may also be solved by means of the formulæ for right-angled trihedrals, by conceiving a plane to be drawn through one of the edges at right angles to the opposite side, which will divide the trihedral into right-angled trihedrals. The mode of proceeding for this purpose can offer no difficulty to the student. When only one part is required to be determined, the computation by this method will be found somewhat shorter.

196. Compare the first four cases with the corresponding cases of oblique-angled triangles.

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## LESSON XIX.

Expressions for determining the Spherical Excess—Measure of a trihedral angle—Spherical triangles—Area of a spherical triangle—Examples and Exercises.

### EXPRESSIONS FOR DETERMINING THE SPHERICAL EXCESS.

197. *To find an expression for the sine of half the spherical excess, in function of the three sides.*

Since $\epsilon = A+B+C-180°$,

$$\tfrac{1}{2}\epsilon = \tfrac{1}{2}(A+B+C-180°) = \tfrac{1}{2}(A+B)-(90°-\tfrac{1}{2}C).$$

Hence, developing and substituting for the sines and cosines of $\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}C$, their values (191) in function of the three sides,

$$\sin \tfrac{1}{2}\epsilon = \sin \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}C - \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}C \quad . \quad . \quad (a)$$

$$= \begin{cases} \sin \frac{1}{2}A \sin \frac{1}{2}C \cos \frac{1}{2}B + \sin \frac{1}{2}B \sin \frac{1}{2}C \cos \frac{1}{2}A \\ + \sin \frac{1}{2}A \sin \frac{1}{2}B \cos \frac{1}{2}C - \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C \end{cases}$$

$$= \begin{cases} \dfrac{\sin (s-a) + \sin (s-b) + \sin (s-c) - \sin s}{\sin a \sin b \sin c} \\ \times \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)} \end{cases}$$

$$= \begin{cases} \dfrac{2\sin \frac{1}{2}c \cos \frac{1}{2}(a-b) - 2\sin \frac{1}{2}c \cos \frac{1}{2}(a+b)}{2\sin \frac{1}{2}a \cos \frac{1}{2}a \times 2\sin \frac{1}{2}b \cos \frac{1}{2}b \times 2\sin \frac{1}{2}c \cos \frac{1}{2}c} \\ \times \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)} \end{cases}$$

$$= \frac{\sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}}{2\cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c}:$$

This expression is known as Cagnoli's formula or theorem. It may also be obtained by substituting in ($a$) the values of $\sin\frac{1}{2}(A+B)$ and $\cos\frac{1}{2}(A+B)$ from Delambre's formulæ.

198. *To find an expression for the tangent of the fourth of the spherical excess in function of the three sides.*

From (116) $\tan\frac{1}{2}(\alpha-\beta)=\dfrac{\sin\alpha-\sin\beta}{\cos\alpha+\cos\beta}$.

Let $\alpha=\frac{1}{2}(A+B),\ \beta=90^\circ-\frac{1}{2}C,$

whence $\alpha-\beta=\frac{1}{2}(A+B+C-180^\circ)$.

Then

$$\tan\tfrac{1}{4}\epsilon=\frac{\sin\frac{1}{2}(A+B)-\cos\frac{1}{2}C}{\cos\frac{1}{2}(A+B)+\sin\frac{1}{2}C}\quad\ldots\ldots\ldots\quad(a)$$

$$=\frac{\sin\frac{1}{2}A\cos\frac{1}{2}B+\sin\frac{1}{2}B\cos\frac{1}{2}A-\cos\frac{1}{2}C}{\cos\frac{1}{2}A\cos\frac{1}{2}B-\sin\frac{1}{2}A\sin\frac{1}{2}B+\sin\frac{1}{2}C}$$

$$=\frac{\sin(s-a)+\sin(s-b)-\sin c}{\sin s-\sin(s-c)+\sin c}\sqrt{\frac{\sin s\sin(s-c)}{\sin(s-a)\sin(s-b)}}$$

$$=\frac{2\sin\frac{1}{2}c[\cos\frac{1}{2}(a-b)-\cos\frac{1}{2}c]}{2\sin\frac{1}{2}c(\cos\frac{1}{2}(a+b)+\cos\frac{1}{2}c)}\sqrt{\frac{\sin s\sin(s-c)}{\sin(s-a)\sin(s-b)}}$$

$$=\frac{\sin\frac{1}{4}(b+c-a)\sin\frac{1}{4}(a+c-b)}{\cos\frac{1}{4}(a+b+c)\cos\frac{1}{4}(a+b-c)}\sqrt{\frac{\sin s\sin(s-c)}{\sin(s-a)\sin(s-b)}}$$

$$=\sqrt{\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)}.$$

This formula is due to Lhuilier.

The substitution in ($a$) of the values of $\sin\frac{1}{2}(A+B)$ and $\cos\frac{1}{2}(A+B)$ from Delambre's formulæ (174) would lead to the same result.

199. Other expressions for angular functions of the spherical excess are sometimes used.

Thus $$\cos\tfrac{1}{2}\epsilon=\frac{\cos^2\frac{1}{2}a+\cos^2\frac{1}{2}b+\cos^2\frac{1}{2}c-1}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}$$

$$\cot\tfrac{1}{2}\epsilon=\frac{\cot\frac{1}{2}a\cot\frac{1}{2}b+\cos C}{\sin C}.$$

### MEASURE OF A TRIHEDRAL ANGLE.

200. When two planes (MN and PQ) meet, there are formed upon the same side of one of these planes (MN) two dihedral angles which are supplements the one of the other, and which may be made to vary from zero to two right angles by the revolution of the second plane (PQ) on the common intersection (AA′) of the two planes.

201. If a third plane (XY) be drawn at right angles to this common intersection (AA′), two equal bi-rectangular trihedrals shall be formed in each of these dihedrals, having for their oblique angle the dihedral angle NA′AP, or its supplement MAA′P.

While the dihedral angle increases from zero to two right angles, the bi-rectangular trihedral of which it is the oblique angle, increases continuously from zero to a tri-rectangular trihedral, and then to two tri-rectangular trihedrals; and whatever part or parts the dihedral is of a right angle, the same part or parts of a tri-rectangular trihedral is the corresponding bi-rectangular trihedral.

If, therefore, the tri-rectangular trihedral be taken as the unit of measure for bi-rectangular trihedrals, the same number that expresses the measure of a dihedral referred to the right angle as unit of measure of dihedrals, shall express the measure of the bi-rectangular trihedral that has the dihedral for its oblique angle.

Thus, if a dihedral be an angle of 35°, the corresponding bi-rectangular trihedral will be the $\frac{35}{90}$, or $\frac{7}{18}$ of a tri-rectangular trihedral.

202. Let three planes, MN, PQ, RV, meet in a point S; AA′ be the common intersection of MN and PQ, BB′ of MN and RV, and CC′ of PQ and RV. On the same side of one of the planes, MN for instance, there will be formed the four trihedrals SABC, SA′BC, SB′AC, SA′B′C.

The trihedrals SABC and SA′BC are, together, equal to twice the bi-rectangular trihedral whose oblique angle is A; SABC and SAB′C are together double of the bi-rectangular trihedral whose oblique angle is B; SABC and SA′B′C are equal to SA′B′C′ (which is vertically opposite to SABC on the other side of the plane MN, and, therefore, equal to it), and SA′B′C; and these latter are together double of the bi-rectangular trihedral having C for its oblique angle.

Let T represent a trihedral SABC, $T_A''$, $T_B''$, $T_C''$, the bi-rectangular trihedrals whose oblique angles are A, B, C, respectively; and T‴ a tri-rectangular trihedral.

Then,

$$\begin{aligned} SABC &= T \\ SA'BC &= 2T_A'' - T \\ SAB'C &= 2T_B'' - T \\ SA'B'C &= 2T_C'' - T, \end{aligned}$$

whence,

$$SABC + SA'BC + SAB'C + SA'B'C = 2(T_A'' + T_B'' + T_C'' - T).$$

But the four trihedrals are also together equal to four tri-rectangular trihedrals; therefore,

$$T_A'' + T_B'' + T_C'' - T = 2T''', \text{ or } T = T_A'' + T_B'' + T_C'' - 2T'''.$$

Therefore, $$\frac{T}{T'''} = \frac{T_A''}{T'''} + \frac{T_B''}{T'''} + \frac{T_C''}{T'''} - 2.$$

That is to say, the measure of a trihedral angle referred to the tri-rectangular trihedral as standard, is equal to the excess above twice the measure of the tri-rectangular trihedral of the sum of the measures of the bi-rectangular trihedrals, whose oblique angles are respectively the same as the dihedrals of the trihedral.

203. As the measures of a bi-rectangular trihedral and of its oblique angle, referred respectively to the tri-rectangular trihedral and a right angle, are expressed by the same number, it may, therefore, be said that a trihedral has for its measure the excess of its three angles above two right angles; in other words, whatever part or parts the spherical excess of a given trihedral is of a right angle, the same part or parts will the given trihedral be of a tri-rectangular trihedral.

For instance, if 73° 15′, 121° 43′, 60° 54′, be the three angles of a trihedral, its spherical excess will be 75° 52′, and its measure will be expressed by this number 75° 52′, if the tri-rectangular trihedral be expressed by 90°. It will, in other words, be the $\frac{569}{675}$ or ·8427 of a tri-rectangular trihedral.

### SPHERICAL TRIANGLES.

204. If a sphere have its centre at the vertex of a trihedral angle, the three planes forming this angle, produced if necessary, shall determine by their intersections with the spherical surface three arcs of great circles meeting two and two. The portion of the surface enclosed by these three arcs is called a *spherical triangle*, of which the arcs are the sides, and the dihedral angles of the trihedral are the angles.

205. Each of the arcs thus determined on the spherical surface, that is, each of the sides of the spherical triangle, has for its measure, whether expressed in degrees, minutes, and seconds, or in circular measure, the same number as expresses the measure of the plane angle forming the corresponding side of the trihedral.

The various relations already established, connecting together the angular functions of the angles and sides of a trihedral angle, may, therefore, be transferred without any alteration to the angles and sides of the corresponding spherical triangle.

206. All the foregoing problems and formulæ, derived from the properties of the trihedral angle, apply equally to the spherical triangle: and if placed before the student as referring to a spherical triangle, he will have no other alteration to make in the investigation than to change the term *trihedral* or *trihedral angle* into *spherical triangle*, and to substitute the word *angle* for the term *dihedral* or *dihedral angle*.

207. If at the centre of the sphere to which a spherical triangle belongs, radii be drawn at right angles to the sides of the corresponding trihedral, they will determine a new trihedral supplementary to the former (152), and whose sides will form by their intersection with the spherical surface a second spherical triangle, such that its angles and sides are respectively the supplements of the sides and angles of the given triangle. With reference to this given triangle, the new one is called the *polar triangle*, and sometimes the *supplementary triangle*.

## AREA OF A SPHERICAL TRIANGLE.

208. Three planes at right angles to one another, passing through the centre of a sphere, divide the whole spherical surface into eight equal parts. If $r$ be the radius of the sphere, each of these tri-rectangular spherical triangles will have for its measure $\frac{1}{2}\pi r^2$ (for the surface of the sphere has for its measure the number $4\pi r^2$).

Now, a spherical triangle will be the same part or parts of the eighth of the spherical surface as the corresponding trihedral at the centre is of the tri-rectangular trihedral; that is, as its spherical excess is of a right angle. Hence, the area of a spherical triangle will be found by multiplying $\frac{1}{2}\pi r^2$ by the quotient of the measure of its spherical excess by the measure of a right angle.

## EXAMPLES.

EXAMPLE I.—*In a right-angled trihedral, one of the sides is 42° 12′, and the opposite angle 48°: to find the remaining parts.*

This belongs to the first part of the Second Case.

*Formulæ.*

L sin $c$ = L sin 42° 12′ + $c$L sin 48°
L sin $b$ = L tan 42° 12′ + L cot 48° — 10
L sin B = L cos 48° + $c$L cos 42° 12′

*Computation for* c.

| | |
|---|---|
| L sin 42° 12′, . . . | 9·8271887 |
| $c$L sin 48°, . . . | 0·1289265 |
| 64° 41′ or 115 19 . . | 9·9561152 |

*Computation for* b.

| | |
|---|---|
| L tan 42° 12′, . . . | 9·9574860 |
| L cot 48°, . . . | 9·9544374 |
| —10, | |
| 54° 44′ or 125 16 . . | 9·9119224 |

*Computation for* B.

| | |
|---|---|
| L cos 48°, . . . | 9·8255109 |
| $c$L cos 42° 12′, . . | 0·1302963 |
| 64° 36′ or 115 24 . . | 9·9558072 |

The problem admits two solutions, and only two; for the three acute values, or the three obtuse values, must be taken together.

EXAMPLE II.—*To reduce the angle subtended by two places not in the horizontal plane to that in the horizontal plane.*

Let the angles of elevation or of depression of two objects A and B at a given station O be observed, and also the angle AOB which the two objects subtend at the station.

The two vertical planes through the station and each of the objects together with the plane AOB, determine a trihedral in which the three sides are the complements of the angles of elevation or of depression and the angle subtended at the station by the two places. These sides being known, the angle contained by the vertical planes through the objects, which is equal to the horizontal angle corresponding to the angle AOB, can be computed.

Let 47° 45′ 39″ be the angle subtended at the station by the two places, and 20° 10′ 41″, 9° 42′ 24″, their angles of elevation.

| | | | | | | |
|---|---|---|---|---|---|---|
| | 90° | | | 90° | | |
| | 20 | 10′ | 41″ | 9 | 42′ | 24″ |
| | 69 | 49 | 19 | 80 | 17 | 36 |
| | 69° | 49′ | 19″ | 29° | 6′ | 58″ |
| | 80 | 17 | 36 | 18 | 38 | 41 |
| | 47 | 45 | 39 | | | |
| 2) | 197 | 52 | 34 | | | |
| | 98 | 56 | 17 | | | |

| | | |
|---|---|---|
| L sin 29° 6′, . . . | | 9·6869359 |
| 58″, . . | | 2192 |
| L sin 18° 38′, . . . | | 9·5044853 |
| 41″, . . | | 2558 |
| $c$L sin 60° 49′, | 9·9724775 | |
| 19″, | 148 | |
| | | ·0275077 |
| $c$L sin 80° 17, | 9·9937247 | |
| 36″, | 129 | |
| | | ·0062624 |
| | | 2) 19·2256663 |
| L sin ½ horizontal angle, | | 9·6128331 |
| L sin 24° 12′, . . | | 9·6127023 |
| 27″ . . | | 1308 |
| 48° 24′ 54″, the required angle. | | |

EXAMPLE III.—*Given the latitudes and longitudes of two places on the terrestrial globe, to find their distance.*

Let both the places A and B be situated in the northern hemisphere and on the same side of the first meridian. Let P be the pole.

The great circle connecting the two places and the meridian circles passing through them form a spherical triangle APB in which the two sides PA, PB, are the complements of the latitudes, and the angle APB has for its measure the difference of the longitudes. The third side AB can therefore be found.

If the two places were on opposite sides of the first meridian, their angle of position APB would have for its measure the sum of the longitudes.

What is the distance from Tory Island, lat. 55° 17′ N., long. 8° 16′ W., to South Cape, Spitzbergen, lat. 76° 32′ N., long. 13° 45′ E.?

Let C be the angle of the two meridians, $a$ and $b$ the polar distances of Tory Island and of South Cape respectively. The formulæ will be

$$L\tan\tfrac{1}{2}(A+B)=L\cot 11^\circ 0'\tfrac{1}{2}+L\cos 10^\circ 37'\tfrac{1}{2}+cL\cos 24^\circ 5'\tfrac{1}{2}-10,$$

$$L\tan\tfrac{1}{2}(A-B)=L\cot 11^\circ 0'\tfrac{1}{2}+L\sin 10^\circ 37'\tfrac{1}{2}+cL\sin 24^\circ 5'\tfrac{1}{2}-10,$$

$$L\tan\tfrac{1}{2}c=L\tan 24^\circ 5'\tfrac{1}{2}+L\cos\tfrac{1}{2}(A+B)+cL\cos\tfrac{1}{2}(A-B)-10.$$

| | | |
|---|---|---|
| 13° 45′ | 90° 0′ | 90° 0′ |
| 8 16 | 55 17 | 76 32 |
| 2) 22 1 | 34 43 | 13 28 |
| 11 0½ | 13 28 | 34 43 |
| | 2) 21 15 | 2) 48 11 |
| | 10 37½ | 24 5½ |

*Computation for ½(A+B).*

| | |
|---|---|
| L cot 11° 0′½, . . . | 10·7116797 |
| L cos 10° 37′½, . . | 9·9938640 |
| cL cos 24° 5′½, . . | 0·0395848 |
| —10, | |
| 79° 48′, . . | 10·7451185 |

*Computation for ½(A—B).*

| | |
|---|---|
| L cot 11° 0′½, . . . | 10·7116797 |
| L sin 10° 37′½, . . | 9·2232334 |
| cL sin 24° 5′½, . . | 0·3891295 |
| —10, | |
| 64° 28′, . . | 10·3240426 |

*Computation for c.*

| | |
|---|---|
| L tan 24° 5′½, . . . | 9·6502809 |
| L cos 79° 48′, . . . | 9·2481811 |
| cL cos 64° 38′, . . . | 0·3680077 |
| —10, | |
| 10° 28′, . . | 9·2664697 |

## EXERCISES.

1. The hypotenuse of a right-angled trihedral is 120° 15′ 47″, and one of the sides 97° 19′ 38″.

2. An angle of a right-angled trihedral measures 59° 48′ 23″ and the adjacent side 159° 17′ 52″.

3. The two angles of a right-angled trihedral are 99° 17′ 36″, and 114° 37′ 24″; to find the hypotenuse and the sides.

4. In a quadrantal trihedral the angle opposite to the right-angled side is 72° 48′, and one of the other angles 43° 15′: to find the remaining parts.

5. One of the sides of a quadrantal trihedral is 29° 36′, and the opposite angle 63° 18′: it is required to compute the remaining parts.

6. The two sides of a spherical triangle are 48° 25′ 30″, 75° 20′ 10″, and the contained angle 41° 18′ 20″.

7. The three sides of a trihedral angle are 72° 20′ 20″, 123° 36′ 40″, 60° 25′ 20″.

8. The two sides of a trihedral are 71° 28′, 159° 14′, and the angle opposite to the former side is 100° 45′.

9. The two sides of a spherical triangle are 75° 35′, 50° 18′, and the angle opposite to the latter is 63° 12′.

10. The two angles of a spherical triangle are 80° 19′ 28″, 120° 29′ 14″, and the side opposite to the latter is 98° 48′.

11. The two angles of a spherical triangle are 48° 15′ and 73° 29′, and the adjacent side is 67° 42′.

12. The three sides of a trihedral are 58° 14′ 42″, 69° 45′ 15″, and 43° 37′ 28″: to compute the spherical excess.

13. The angles of elevation of two places above a station are 37° 48′ 50″ and 48° 49′ 20″; and they subtend at the station an angle of 15° 15′ 45″. Find the horizontal angle between the two places.

14. Two objects, not in the same horizontal plane, subtend at the top of a steeple an angle of 49° 18′; and their angles of depression are 12° 25′, and 32° 46′. What horizontal angle do these objects subtend?

15. What is the distance in geographical miles between Dublin and Paris? The latitudes are 53° 23′ 13″ N., and 48° 50′ 13″ N.; and the longitudes 6° 21′ 30″ W., and 2° 20′ 24″ E.

16. Find the distance from Dublin to Mauritius in lat. 20° 10′ S. and long. 57° 28′ E.

17. What are the distances from Cork and Liverpool to New York? the latitudes being 51° 48′ N., 53° 24′ 40″ N., 40° 42′ 6″ N.; and the longitudes 8° 14′ 30″ W., 2° 58′ 55″ W., 73° 59′ W.

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Dublin: ALEX. THOM & SONS, 87 & 88, Abbey-street.

# ERRATA.

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Page 26, 10th line, *for* $\left(\frac{BE}{AD}\right)^2$, *read* $\frac{BE}{AD}$.

Page 31, 4th line, *for* $\cot^2 A + \cos^2 A$, *read* $\cot^2 A - \cos^2 A$.

" " 22nd line, *for* $= \cos(a+45^\circ)\sqrt{2}$, *read* $= -\cos(a+45^\circ)\sqrt{2}$.

" " 3rd line from the bottom, *for* $\cos^2 B$, *read* $\sin^2 B$.

Page 32, 17th line, *for* $\sin 18^\circ + \sin 54^\circ = \sin 30^\circ$,
*read* $\sin 18^\circ = \sin 54^\circ - \sin 30^\circ$.

Page 116, 14th line, *for* angle, *read* angles.